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Model and control of multiagent systems based on idempotent algebra methods

Institute of Control Sciences
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The goal

The main goal of our work is developing and investigating of a new class of single agent and multiagent motion models based on idempotent algebra methods.

Assumptions

Dynamic	Environment is able to change while agent moving through it.
Uncertain	Agents don't know exactly how the surrounding world works.
Unboundedly uncertain	Agents have no a priori information about uncertain parameters.
Fully observable	Full information about environment is available to agent in every moment.
Greedy agents	Agents, which always choose suboptimal actions.

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Плакат 3

Single agent systems

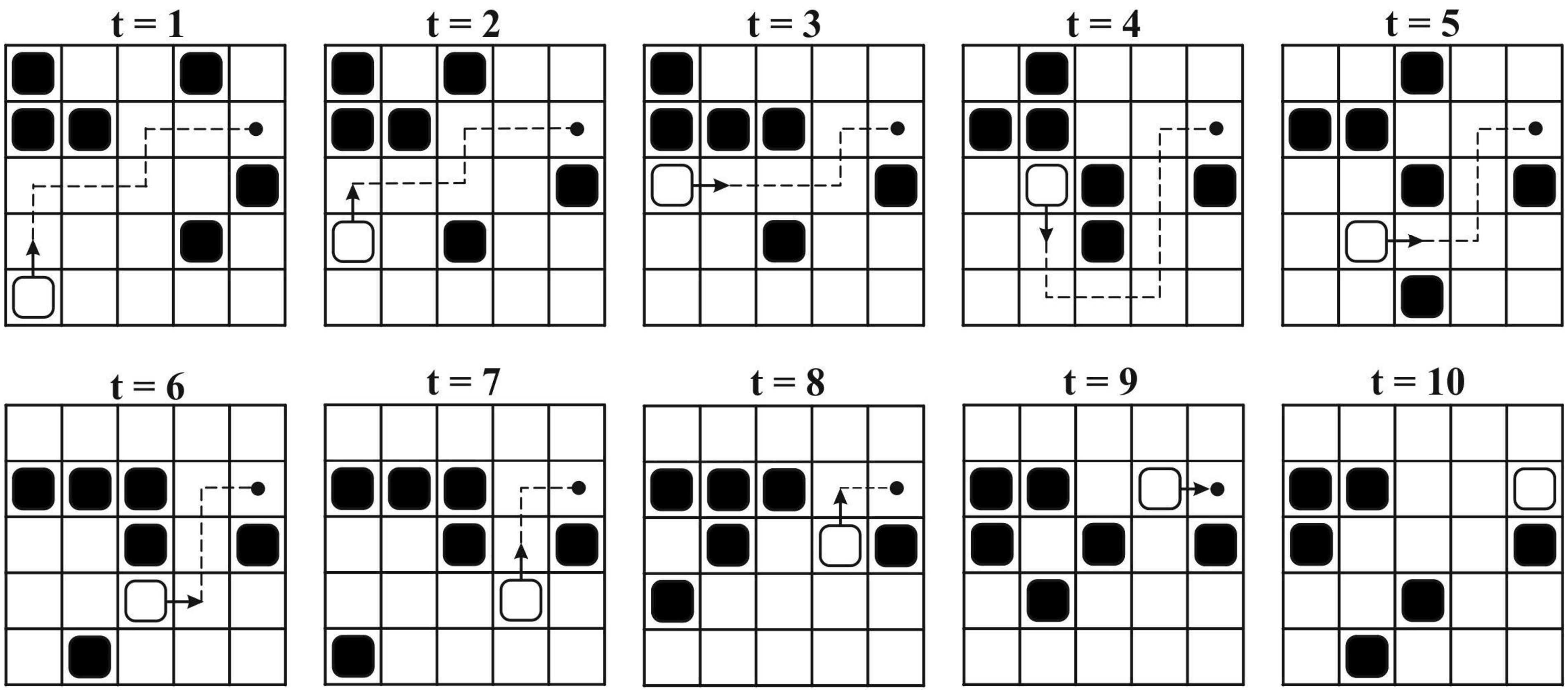

Fig. 1. Single agent motion

Multiagent systems

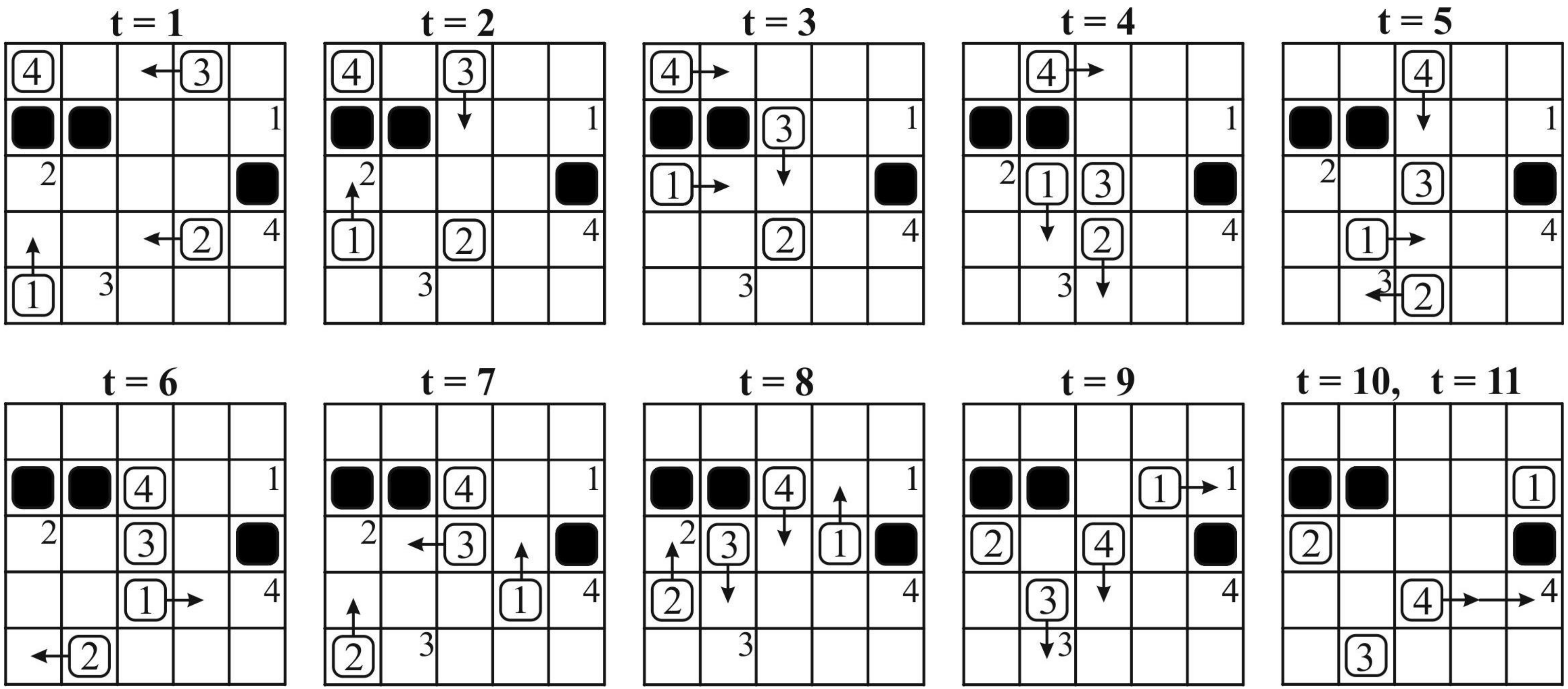

Fig. 2. Multiagent motion

Idempotent fusion semirings

Idempotent fusion semiring $\boldsymbol{F}_{\min}$ is an algebraic structure $\langle \boldsymbol{F}, \oplus, \odot, \boldsymbol{0}, \boldsymbol{1} \rangle$ with idempotent addition и fusion product in the role of multiplication.

$$\boldsymbol{F} = \boldsymbol{N}^{\star} \cup \{\chi\} = \bigcup_{k=0}^{+\infty} \boldsymbol{N}^k \cup \{\chi\},\ \ a \oplus b = \operatorname{lexmin}\left(\operatorname*{argmin}_{z \in \{a,b\}} |z|\right),\ \ a \odot b = \begin{cases} \alpha_1 \dots \alpha_k \beta_2 \dots \beta_l, \\ \emptyset,\ \ otherwise; \end{cases} \quad (1)$$

$$\boldsymbol{0} = \emptyset,\quad \boldsymbol{1} = \chi,\ \ |\emptyset| = +\infty,\ \ |\chi| = 0;$$

$\oplus$	$\emptyset$	χ	1	2	3	1-2	1-3	2-1	2-3	3-1	3-2
$\emptyset$	$\emptyset$	χ	1	2	3	1-2	1-3	2-1	2-3	3-1	3-2
χ	χ	χ	χ	χ	χ	χ	χ	χ	χ	χ	χ
1	1	χ	1	1	1	1	1	1	1	1	1
2	2	χ	1	2	2	2	2	2	2	2	2
3	3	χ	1	2	3	3	3	3	3	3	3
1-2	1-2	χ	1	2	3	1-2	1-2	1-2	1-2	1-2	1-2
1-3	1-3	χ	1	2	3	1-2	1-3	1-3	1-3	1-3	1-3
2-1	2-1	χ	1	2	3	1-2	1-3	2-1	2-1	2-1	2-1
2-3	2-3	χ	1	2	3	1-2	1-3	2-1	2-3	2-3	2-3
3-1	3-1	χ	1	2	3	1-2	1-3	2-1	2-3	3-1	3-1
3-2	3-2	χ	1	2	3	1-2	1-3	2-1	2-3	3-1	3-2

Fig. 3. Cayley table for addition

Cayley tables

$\oplus$	$\emptyset$	χ	1	2	3	1-2	1-3	2-1	2-3	3-1	3-2
$\emptyset$	$\emptyset$	χ	1	2	3	1-2	1-3	2-1	2-3	3-1	3-2
χ	χ	χ	χ	χ	χ	χ	χ	χ	χ	χ	χ
1	1	χ	1	1	1	1	1	1	1	1	1
2	2	χ	1	2	2	2	2	2	2	2	2
3	3	χ	1	2	3	3	3	3	3	3	3
1-2	1-2	χ	1	2	3	1-2	1-2	1-2	1-2	1-2	1-2
1-3	1-3	χ	1	2	3	1-2	1-3	1-3	1-3	1-3	1-3
2-1	2-1	χ	1	2	3	1-2	1-3	2-1	2-1	2-1	2-1
2-3	2-3	χ	1	2	3	1-2	1-3	2-1	2-3	2-3	2-3
3-1	3-1	χ	1	2	3	1-2	1-3	2-1	2-3	3-1	3-1
3-2	3-2	χ	1	2	3	1-2	1-3	2-1	2-3	3-1	3-2

Fig. 4. Cayley table for multiplication

Basic linear algebra notions

For matrices $A=[a_{ij}]\in\boldsymbol{F}^{m\times n}$, $B=[b_{ij}]\in\boldsymbol{F}^{m\times n}$, $C=[c_{ij}]\in\boldsymbol{F}^{n\times l}$ ***addition and multiplication*** *are defined conventionally as follows:*

$$\{A\oplus B\}_{ij}=a_{ij}\oplus b_{ij},\ \ \{BC\}_{ij}=\bigoplus_{k=1}^{n}b_{ik}c_{kj}. \tag{2}$$

For square matrix $A\in\boldsymbol{F}^{n\times n}$ ***the Kleene operator*** is the following power series

$$A^{\star}=\bigoplus_{k=0}^{+\infty}A^k=I\oplus A\oplus A^2\oplus\ldots\oplus A^k\oplus\ldots \tag{3}$$

For matrices $A,B\in\boldsymbol{F}^{m\times n}$ ***the Hadamard product*** $\circledast$ and for vectors $x\in\boldsymbol{F}^m, y\in\boldsymbol{F}^n$ ***outer product*** are defined by formulas

$$\{A\circledast B\}_{ij}=a_{ij}b_{ij},\ \ \{xy^T\}_{ij}=x_iy_j. \tag{4}$$

A vectorization operator $\mathrm{v}:\boldsymbol{F}^{m\times n}\to\boldsymbol{F}^{mn}$, also known as vec, stacks columns $a_1,a_2,\ldots,a_n\in\boldsymbol{F}^m$ of matrix $A=[a_1,a_2,\ldots,a_n]\in\boldsymbol{F}^{m\times n}$ below one another

$$A^{\mathrm{v}}=\mathrm{vec}A=\begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix}.$$

Introduced algebraic notions

A binarization operator b turns sqalar $a = \alpha_1 \dots \alpha_k \in \boldsymbol{F}$ into binary vector of size n with identities on α_1th, ..., α_kth places and zoroes on others.

Function $\underset{i,j}{\mathrm{slice}}: \boldsymbol{F} \to \boldsymbol{F}$, picks out a subword from ith to jth symbol of the argument word $a = \alpha_1 \dots \alpha_k \in \boldsymbol{F}$ inclusively and her particular case when $i = j$ ***function*** $\mathrm{pop}_{\mathrm{i}}: \boldsymbol{F} \to \boldsymbol{F}$ picks out only one symbol. This functions are formally defined by formulas

$$\underset{[i,j]}{\mathrm{slice}}(\mathrm{a}) = \begin{cases} \alpha_i \dots \alpha_j, & if\ i < j < k; \\ \alpha_i \dots \alpha_k, & if\ i < k < j; \\ \alpha_k, & if\ k < i < j. \end{cases} \qquad \underset{\mathrm{i}}{\mathrm{pop}}(a) = \begin{cases} \alpha_i, & if\ i < k; \\ \alpha_k, & if\ i \geq k. \end{cases} \tag{5}$$

Let's for fixed dimension $n = 9$ take a word $a = 5\text{-}9\text{-}10\text{-}1\text{-}4\text{-}6\text{-}7\text{-}12\text{-}15$ and indices $i = 4,\ \ j = 7$. Then the following identity holds on

$$\mathrm{slice}^{\mathrm{b}}_{4,7}\left(5\text{-}9\text{-}10\text{-}\underline{1\text{-}4\text{-}6\text{-}7}\text{-}12\text{-}15\right) = (1\text{-}4\text{-}6\text{-}7)^{\mathrm{b}} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix}^{\mathrm{v}}. \tag{6}$$

Algebraic model of single agent motion

Motion of single greedy agent motion is described by following nonlinear dynamical system over idempotent fusion semiring $\boldsymbol{F}_{\min}$

$$\begin{cases} u[t] = \bar{\theta}[t], \\ x[t] = \operatorname*{pop}^{\mathrm{b}}_{1+\mathrm{c}[\mathrm{t}]}\left(x^{\mathrm{T}}[t]\left(u[t]u^{\mathrm{T}}[t] \circledast A[t]\right)^{\star} g[t]\right), \\ u[t] = \operatorname*{slice}^{\mathrm{b}}_{1+\mathrm{c}[\mathrm{t}]}\left(x^{\mathrm{T}}[t]\left(u[t]u^{\mathrm{T}}[t] \circledast A[t]\right)^{\star} g[t]\right), \\ x[0] = x_0, \quad t \in \boldsymbol{N}, \end{cases} \tag{7}$$

where $\theta[t]$ – environment state, $u[t]$ – control of the agent at moment t, $x[t]$ – state of the agent at moment t, $y[t]$ – output of the agent at moment t, $g[t]$ – goal of the agent at moment t, $c[t]$ – speed of the agent at moment t, $A[t]$ – basic actions matrix of the agent at moment t, $\bar{\theta}$ – the negation operator, $\star$ – the Kleene operator, $\circledast$ – the Hadamard product, $\boldsymbol{N}$ – the natural numbers set.

Communication graph

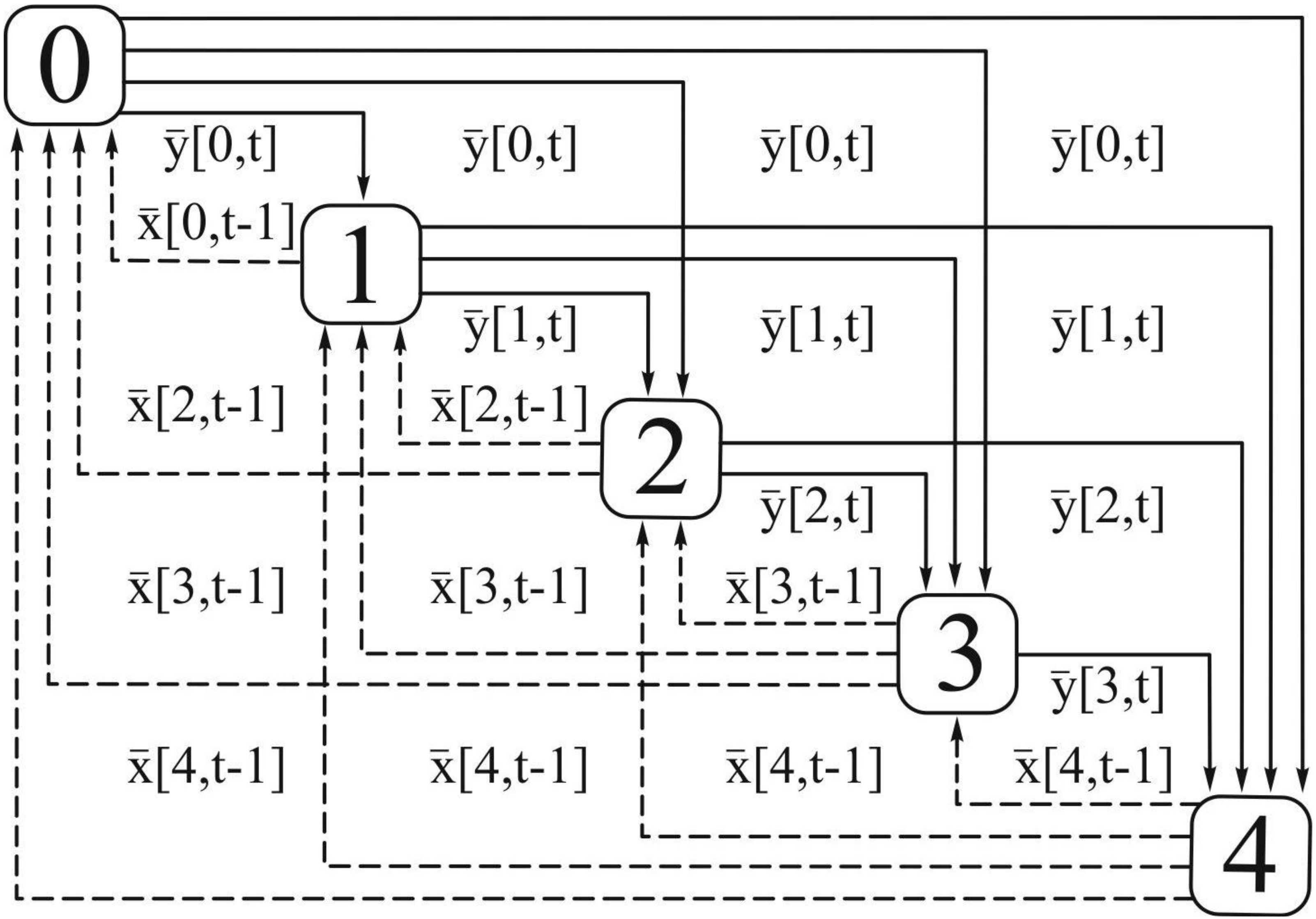

Fig. 5. Communication graph for nature and four agents

Communication matrix for nature and four agents

$$M_C[t] = \begin{bmatrix} O & \bar{y}[0,t] & \bar{y}[0,t] & \bar{y}[0,t] & \bar{y}[0,t] \\ \bar{x}[1,t-1] & O & \bar{y}[1,t] & \bar{y}[1,t] & \bar{y}[1,t] \\ \bar{x}[2,t-1] & \bar{x}[2,t-1] & O & \bar{y}[2,t] & \bar{y}[2,t] \\ \bar{x}[3,t-1] & \bar{x}[3,t-1] & \bar{x}[3,t-1] & O & \bar{y}[3,t] \\ \bar{x}[4,t-1] & \bar{x}[4,t-1] & \bar{x}[4,t-1] & \bar{x}[4,t-1] & O \end{bmatrix},$$

$$\begin{cases} u[1,t] & = & \bar{y}[0,t]\oplus\bar{x}[2,t-1]\oplus\bar{x}[3,t-1]\oplus\bar{x}[4,t-1], \\ u[2,t] & = & \bar{y}[0,t]\oplus\bar{y}[1,t]\oplus\bar{x}[3,t-1]\oplus\bar{x}[4,t-1], \\ u[3,t] & = & \bar{y}[0,t]\oplus\bar{y}[1,t]\oplus\bar{y}[2,t]\oplus\bar{x}[4,t-1], \\ u[4,t] & = & \bar{y}[0,t]\oplus\bar{y}[1,t]\oplus\bar{y}[2,t]\oplus\bar{y}[3,t]. \end{cases}$$

In general form control $u[r,t]$ is defined by formula

$$u[r,t] = \bigoplus_{s=0}^{r-1} \bar{y}[s,t] \oplus \bigoplus_{s=r+1}^{p} \bar{x}[s,t-1].$$

Algebraic model of multiagent motion

Motion of sequentially interacting greedy agent groups is described by nonlinear 2D-system over idempotent fusion semiring $\boldsymbol{F}_{\min}$.

$$\begin{cases} u[r,t] = \bigoplus_{s=0}^{r-1} \bar{y}[s,t] \oplus \bigoplus_{s=r+1}^{p} \bar{x}[s,t-1], \\ x[r,t] = \underset{1+\mathrm{c}[\mathrm{t}]}{\mathrm{pop}^{\mathrm{b}}} \left(x^{\mathrm{T}}[r,t] \left(u[r,t] u^{\mathrm{T}}[r,t] \circledast A[r,t] \right)^{\star} g[r,t] \right), \\ u[r,t] = \underset{1+\mathrm{c}[\mathrm{t}]}{\mathrm{slice}^{\mathrm{b}}} \left(x^{\mathrm{T}}[r,t] \left(u[r,t] u^{\mathrm{T}}[r,t] \circledast A[r,t] \right)^{\star} g[r,t] \right), \\ x[r,0] = x_0[r], \;\; y[0,t] = y_0[t], \;\; r \in \boldsymbol{Z}_{1+p}, \;\; t \in \boldsymbol{N}, \end{cases} \tag{8}$$

where $u[r,t]$ – control of agent r at moment t, $x[r,t]$ – state of agent r at moment t, $y[r,t]$ – goal of agent r at moment t, $g[r,t]$ – goal of agent r at moment t, $c[r,t]$ – speed of agent r at moment t, $A[r,t]$ – basic actions matrix of agent r at moment t, $\bar{y}$ – the negation operator, $\star$ – the Kleene operator, $\circledast$ – the Hadamard product, $\boldsymbol{N}$ – the natural numbers set, $\boldsymbol{Z}_{1+p}$ – ring of integers modulo $1+p$, p – number of agents.