Numerical methods of seeking optimistic solutions in nonlinear bilevel optimization problems

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Problems of bilevel optimization

General problem formulation

$$\left.\begin{array}{l} F(x,y) \downarrow \min\limits_{x,y}, \quad (x,y) \in D \subset \mathbf{R}^{m+n}, \quad y \in Y_*(x), \\ Y_*(x) \stackrel{\triangle}{=} \operatorname{Arg}\min\limits_{y}\{G(x,y) \mid (x,y) \in D_1 \subset \mathbf{R}^{m+n}\}. \end{array}\right\} \quad (\mathcal{BP})$$

$(\mathbf{y_j}, \quad \mathbf{j} = \mathbf{1}, \ldots, \mathbf{n}$ **do not depend on each other** $)$

Quadratic-linear bilevel optimization problem

$$\left.\begin{array}{l} F(x,y) \stackrel{\triangle}{=} \frac{1}{2}\langle x, Cx\rangle + \langle c, x\rangle + \frac{1}{2}\langle y, C_1 y\rangle + \langle c_1, y\rangle \downarrow \min\limits_{x,y}, \\ (x,y) \in D \stackrel{\triangle}{=} \{x \in \mathbf{R}^m, y \in \mathbf{R}^n \mid Ax + By \leq a, \ x \geq 0\}, \\ y \in Y_*(x) \stackrel{\triangle}{=} \operatorname{Arg}\min\limits_{y}\{\langle d, y\rangle \mid A_1 x + B_1 y \leq b, \ y \geq 0\}, \end{array}\right\} \quad (\mathcal{BP}_{QL})$$

where $C = C^T \geq 0, \ C_1 = C_1^T \geq 0.$

The elaborated approach to problem $(\mathcal{BP}_{QL})$

1. Reducing problem $(\mathcal{BP}_{QL})$ to one-level problem with nonconvex feasible set.
2. Reducing the latter problem to a series of problems with nonconvex objective function.
3. Applying Global Search Theory to the reduced problems:
   - a special local search methods, which takes into account the structure of the problem under scrutiny;
   - the global search procedures (based on Global Optimality Conditions proved by A.S. Strekalovsky), which allow to improve the point provided by the local search.

1. A.S. Strekalovsky, A.V. Orlov, A.V. Malyshev. **Local search in a quadratic-linear bilevel programming problem** // Numerical Analysis and Applications. 2010. V.3, No.1, P.59-70.
2. A.S. Strekalovsky, A.V. Orlov, A.V. Malyshev. **On computational search for optimistic solutions in bilevel problems** // Journal of Global Optimization. 2010. V.48, No.1, P.159-172.

Computational simulation (KNITRO & Global Search)

Problem	F_*	F_{KNITRO}	Time	$F_{GlSearch}$	Time
1x1_1	−49	**−49.001**	0.6	**−49**	1.7
2x3_1	−29.2	**−29.2**	0.8	**−29.2**	1.5
2x3_2	−18.4	**−18.4**	0.5	**−18.4**	0.8
5x5_1	−9	**−9**	8.7	**−9**	4.7
5x5_2	−5	**−5**	9.9	**−5**	4.6
10x10_1	−38	−30	1:17.2	**−38**	23.7
10x10_2	−26	**−26**	1:17.0	**−26**	12.4
15x15_1	−19	**−19**	11:08.5	**−19**	19.0
15x15_2	−27	−19	5:43.6	**−27**	31.5
20x20_1	−24	**−24**	19:13.7	**−24**	26.9
20x20_2	−48	**−48**	30:20.5	**−47.999**	1:10.9
20x20_3	−52	−32	29:54.4	**−52**	52.0
30x30_1	−142	−134	1:34:34.9	**−141.997**	1:46.8
30x30_2	−58	−38	1:31:23.5	**−58**	1:26.2
30x30_3	−42	−29.999	2:39:15.4	**−42**	51.5

LP & QP — Xpress; CPU: Intel Core 2 Quad 2.8GHz, Intel Core 2 Duo 2.0GHz.

D.C.-quadratic bilevel problem

$$\left.\begin{array}{c} F(x,y) \overset{\triangle}{=} g(x,y) - h(x,y) \downarrow \min\limits_{x,y}, \\ (x,y) \in X \overset{\triangle}{=} \{(x,y) \in \boldsymbol{R}^{m+n} \mid f_l(x,y) \leq 0, l = 1,\ldots,p\}, \\ y \in Y_*(x) \overset{\triangle}{=} \operatorname{Arg}\min\limits_{y}\{\frac{1}{2}\langle y, Cy\rangle + \langle x, Qy\rangle + \langle d, y\rangle \mid \\ \mid y \in Y(x)\}, \quad Y(x) \overset{\triangle}{=} \{y \in \boldsymbol{R}^n \mid Ax + By \leq b\} \end{array}\right\} \qquad (\mathcal{P})$$

$g(\cdot)$, $h(\cdot)$, $f_l(\cdot)$, $l = 1, \ldots, p$ are convex, continuously differentiable functions; $x \in \boldsymbol{R}^m$, d, $y \in \boldsymbol{R}^n$, $b \in \boldsymbol{R}^q$, A, B, C, Q are matrices, $C = C^T \geq 0$.

Problem with a bilinear equality constraint

$$\begin{array}{c} F(x,y) \overset{\triangle}{=} g(x,y) - h(x,y) \downarrow \min\limits_{x,y,v}, \quad (x,y) \in X, \\ Cy + d + xQ + vB = 0, \quad v \geq 0, \quad Ax + By \leq b, \\ r(x,y,v) \overset{\triangle}{=} \langle v, b - Ax - By\rangle = 0. \end{array} \qquad (\mathcal{P}_1)$$

Reduction of the problem to a single-level problem

Problem with a d.c. goal function

$$\Phi(x,y,v) \stackrel{\triangle}{=} F(x,y) + \mu r(x,y,v) \downarrow \min_{x,y,v},$$

$$(x,y,v) \in D \stackrel{\triangle}{=} \{(x,y,v) \mid (x,y) \in X,\ v \geq 0, \qquad (\mathcal{P}(\mu))$$
$$Cy + d + xQ + vB = 0,\ Ax + By \leq b\},$$

where $\mu > 0$ is a penalty parameter.

Let $(x(\mu), y(\mu), v(\mu))$ be a solution of problem $(\mathcal{P}(\mu))$ corresponding to some fixed parameter $\mu > 0$.

Proposition 1

i) Let for some $\mu = \hat{\mu} > 0$ the equality $r(x(\hat{\mu}), y(\hat{\mu}), v(\hat{\mu})) = 0$ be satisfied for the solution $(x(\hat{\mu}), y(\hat{\mu}), v(\hat{\mu}))$ of problem $(\mathcal{P}(\mu))$. Hence the triplet $(x(\hat{\mu}), y(\hat{\mu}), v(\hat{\mu}))$ turns out to be a solution of problem $(\mathcal{P}_1)$.
ii) For any value of $\mu > \hat{\mu}$ the function $r(x(\hat{\mu}), y(\hat{\mu}), v(\hat{\mu})) = 0$, and $(x(\mu), y(\mu), v(\mu))$ turns out to be a solution of problem $(\mathcal{P}_1)$.

D.C. decompositon of the goal function

$$\Phi(x,y,v) = G(x,y,v) - H(x,y,v),$$

where $H(x,y,v) \stackrel{\triangle}{=} h(x,y) + \frac{\mu}{4}(\|v+Ax\|^2 + \|v+By\|^2),$

$$G(x,y,v) \stackrel{\triangle}{=} g(x,y) + \frac{\mu}{4}(4\langle v,b\rangle + \|v-Ax\|^2 + \|v-By\|^2).$$

Iteration of Special Local Search Method (SLSM)

Let (x_0, y_0, v_0) be a starting point. If we have the triplet (x^s, y^s, v^s), then the triplet $(x^{s+1}, y^{s+1}, v^{s+1})$ is obtained as approximate solution of problem linearized at the point (x^s, y^s, v^s):

$$G(x,y,v) - \langle \nabla_{xy} h(x^s,y^s)(x,y)\rangle - \frac{\mu}{2}(\langle v^s + Ax^s, v\rangle + \langle (v^s + Ax^s)A, x\rangle +$$
$$+\langle v^s + By^s, v\rangle + \langle (v^s + By^s)B, y\rangle) \downarrow \min_{x,y,v}, \quad (x,y,v) \in D. \qquad (\mathcal{PL}_s)$$

So, the triplet $(x^{s+1}, y^{s+1}, v^{s+1})$ satisfies the following inequality:

$$G(x^{s+1}, y^{s+1}, v^{s+1}) - \langle \nabla H(x^s, y^s, v^s), (x^{s+1}, y^{s+1}, v^{s+1})\rangle \leq$$
$$\leq \inf_{x,y,v} \{G(x,y,v) - \langle \nabla H(x^s,y^s,v^s), (x,y,v)\rangle\} + \delta_s. \qquad (1)$$

Local search

Theorem 1

If $\delta_s \geq 0,\ s = 0, 1, 2, ...;\quad \sum_{s=0}^{\infty} \delta_s < +\infty$, then the sequence $\{\Phi_s\}$ of values of function $\Phi_s \stackrel{\triangle}{=} \Phi(x^s, y^s, v^s)$, which is generated by the rule (1), converges. If $(x^s, y^s, v^s) \to (\hat{x}, \hat{y}, \hat{v})$, then the limit point $(\hat{x}, \hat{y}, \hat{v})$ satisfies the following inequality: $G(\hat{x}, \hat{y}, \hat{v}) - \langle \nabla H(\hat{x}, \hat{y}, \hat{v}), (\hat{x}, \hat{y}, \hat{v}) \rangle \leq$

$$\leq G(x, y, v) - \langle \nabla H(\hat{x}, \hat{y}, \hat{v}), (x, y, v) \rangle \quad \forall (x, y, v) \in D. \qquad (2)$$

The point satisfying inequality (2) be called a critical point of the problem $(\mathcal{P}(\mu))$.

Any critical point $(\hat{x}, \hat{y}, \hat{v})$ satisfying classical stationarity conditions:
$\langle \nabla G(\hat{x}, \hat{y}, \hat{v}) - \nabla H(\hat{x}, \hat{y}, \hat{v}), (x, y, v) - (\hat{x}, \hat{y}, \hat{v}) \rangle \geq 0 \quad \forall (x, y, v) \in D.$

Stopping criterion

If the inequality $\qquad \Phi(x^s, y^s, v^s) - \Phi(x^{s+1}, y^{s+1}, v^{s+1}) \leq \tau/2 \qquad (3)$
is satisfied, then the triplet (x^s, y^s, v^s) turns out to be $(\tau/2 + \delta_s)$-critical point of problem $(\mathcal{P}(\mu))$.

Global Search

Global Optimality Conditions (A.S. Strekalovsky)

The feasible point $(x^*, y^*, v^*) \in Sol(\mathcal{P}(\mu))$ iff

$$\forall(\bar{x}, \bar{y}, \bar{v}; \gamma) \in \boldsymbol{R}^{m+n+q+1} : H(\bar{x}, \bar{y}, \bar{v}) = \gamma - \zeta, \;\; \zeta \stackrel{\triangle}{=} \Phi(x^*, y^*, v^*), \quad \text{(A)}$$

$$G(\bar{x}, \bar{y}, \bar{v}) \leq \gamma \leq \sup(G, D), \quad \text{(B)}$$

$$G(x, y, v) - \gamma \geq \langle \nabla H(\bar{x}, \bar{y}, \bar{v}), (x, y, v) - (\bar{x}, \bar{y}, \bar{v}) \rangle \;\; \forall (x, y, v) \in D. \quad \text{(C)}$$

Global Search Procedure

Let there be a known τ-critical point $(\hat{x}, \hat{y}, \hat{v})$ obtained by the SLSM.

1. Construct an approximation $\mathcal{A}$ of the level surface of the convex function $H(x, y, v)$ for some γ.
2. Verify the left inequality from (B) for each point of $\mathcal{A}$.
3. Execute from the selected points the SLSM.
4. Compare the values of the goal function at each obtained critical point to ζ. If some new value is better than ζ turn back stage 1.

Bilevel problem with an equilibrium at the lower level

General problem formulation

$$\left.\begin{array}{c} F(x, y_1, \dots, y_N) \uparrow \max\limits_{x, y_1, \dots, y_N}, \\ x \in X \subset \boldsymbol{R}^m, \quad (y_1, \dots, y_N) \in NE(\Gamma(x))(PE(\Gamma(x))), \end{array}\right\} \quad (\mathcal{BP}_\Gamma)$$

where $NE(\Gamma(x))$ is the set of Nash (Pareto) equilibrium points of the game

$$G_k(x, y_1, \dots, y_N) \uparrow \max_{y_k}, \quad y \in Y_k(x) \subset \boldsymbol{R}^{n_k}, \quad k = 1, \dots, N. \quad (\Gamma(x))$$

$$(\mathbf{y_j}, \quad \mathbf{j = 1, \dots, n} \quad \textbf{depend on each other} \;)$$

Problem $(\mathcal{BP}_\Gamma)$ is the special case of **Mathematical Programs with Equilibrium Constraints** (MPEC).

Bilevel problem with an equilibrium at the lower level

Bilevel problem with a bimatrix game at the lower level

$$\left.\begin{array}{c} \langle c, x\rangle + \langle d_1, y\rangle + \langle d_2, z\rangle \uparrow \max\limits_{x,y,z}, \\ x \in X = \{x \in \boldsymbol{R}^m \mid Ax \leq a,\ x \geq 0,\ \langle b_1, x\rangle + \langle b_2, x\rangle = 1\}, \\ (y, z) \in NE(\Gamma B(x)), \end{array}\right\} \quad (\mathcal{BP}_{\Gamma B})$$

where $NE(\Gamma(x))$ is the set of Nash equilibrium points of the game

$$\left.\begin{array}{c} \langle y, B_1 z\rangle \uparrow \max\limits_{y},\ y \in Y(x) = \{y \in \boldsymbol{R}^{n_1} \mid y \geq 0, \langle e_{n_1}, y\rangle = \langle b_1, x\rangle\}, \\ \langle y, B_2 z\rangle \uparrow \max\limits_{z},\ z \in Z(x) = \{z \in \boldsymbol{R}^{n_2} \mid z \geq 0, \langle e_{n_2}, z\rangle = \langle b_2, x\rangle\}. \end{array}\right\} \quad (\Gamma B(x))$$

$c, b_1, b_2 \in \boldsymbol{R}^m$, $d_1 \in \boldsymbol{R}^{n_1}$; $d_2 \in \boldsymbol{R}^{n_2}$; $a \in \boldsymbol{R}^p$; $b_1 \geq 0$, $b_1 \neq 0$, $b_2 \geq 0$, $b_2 \neq 0$; A, B_1, B_2 are matrices of appropriate dimension; $e_{n_1} = (1, ..., 1)$, $e_{n_2} = (1, ..., 1)$ are vectors of appropriate dimension.

Optimality conditions for generalized bimatrix game

$$\left.\begin{array}{l}\langle y, B_1 z\rangle \uparrow \max\limits_{y}, \;\; y \in Y = \{y \mid y \geq 0, \langle e_{n_1}, y\rangle = \xi_1 > 0\}, \\ \langle y, B_2 z\rangle \uparrow \max\limits_{z}, \;\; z \in Z = \{z \mid z \geq 0, \langle e_{n_2}, z\rangle = \xi_2 > 0\}. \end{array}\right\} \qquad (\Gamma B)$$

Definition

The tuple (y^*, z^*) be called a Nash equilibrium point of the game (ΓB) iff

$$\begin{array}{l}\alpha_* \stackrel{\triangle}{=} \langle y^*, B_1 z^*\rangle \geq \langle y, B_1 z^*\rangle \quad \forall y \in Y, \\ \beta_* \stackrel{\triangle}{=} \langle y^*, B_2 z^*\rangle \geq \langle y^*, B_2 z\rangle \quad \forall z \in Z. \end{array} \qquad (4)$$

Theorem 2

The tuple $(y^*, z^*) \in NE(\Gamma B)$ if and only if there exist numbers α_* and β_* such that the following system is fulfilled:

$$\begin{array}{c}\xi_1(B_1 z^*) \leq \alpha_* e_{n_1}, \;\; \xi_2(y^* B_2) \leq \beta_* e_{n_2}; \;\; \langle y^*, (B_1 + B_2) z^*\rangle = \alpha_* + \beta_*; \\ y^* \geq 0, \;\; \langle e_{n_1}, y\rangle = \xi_1; \quad z^* \geq 0, \;\; \langle e_{n_2}, z\rangle = \xi_2. \end{array} \qquad (5)$$

Nonconvex problem associated with the problem $(\mathcal{BP}_{\Gamma B})$

$$\left.\begin{array}{c}\langle c, x\rangle + \langle d_1, y\rangle + \langle d_2, z\rangle \uparrow \max\limits_{x,y,z}, \\ Ax \leq a, \ x \geq 0, \ \langle b_1, x\rangle + \langle b_2, x\rangle = 1, \\ (y, z) \in NE(\Gamma B(x)), \end{array}\right\} \qquad (\mathcal{BP}_{\Gamma B})$$

$$\left.\begin{array}{lll}\langle y, B_1 z\rangle \uparrow \max\limits_{y}, & y \geq 0, & \langle e_{n_1}, y\rangle = \langle b_1, x\rangle, \\ \langle y, B_2 z\rangle \uparrow \max\limits_{z}, & z \geq 0, & \langle e_{n_2}, z\rangle = \langle b_2, x\rangle. \end{array}\right\} \qquad (\Gamma B(x))$$

Theorem 3

The triplet (x^*, y^*, z^*) is a global optimistic solution of the bilevel problem $(\mathcal{BP}_{\Gamma B})$, if and only if there exist numbers α_* and β_* such that the 5-tuple $(x^*, y^*, z^*, \alpha_*, \beta_*)$ is a global solution of problem $(\mathcal{PB})$.

Nonconvex problem associated with the problem $(\mathcal{BP}_{\Gamma B})$

$$\left.\begin{array}{c} f(x,y,z) \stackrel{\triangle}{=} \langle c,x\rangle + \langle d_1,y\rangle + \langle d_2,z\rangle \uparrow \max\limits_{x,y,z,\alpha,\beta}, \\ Ax \le a, \quad x \ge 0, \quad \langle b_1,x\rangle + \langle b_2,x\rangle = 1, \\ y \ge 0, \quad \langle e_{n_1},y\rangle = \langle b_1,x\rangle, \\ z \ge 0, \quad \langle e_{n_2},z\rangle = \langle b_2,x\rangle, \\ \langle \mathbf{b_1},\mathbf{x}\rangle(\mathbf{B_1 z}) \le \alpha \mathbf{e_{n_1}}, \\ \langle \mathbf{b_2},\mathbf{x}\rangle(\mathbf{y B_2}) \le \beta \mathbf{e_{n_2}}, \\ p(y,z,\alpha,\beta) \stackrel{\triangle}{=} \langle y,(B_1+B_2)z\rangle - \alpha - \beta = 0. \end{array}\right\} \qquad (\mathcal{PB})$$

Theorem 3.

The triplet (x^*, y^*, z^*) is a global optimistic solution of the bilevel problem $(\mathcal{BP}_{\Gamma B})$, if and only if there exist numbers α_* and β_* such that the 5-tuple $(x^*, y^*, z^*, \alpha_*, \beta_*)$ is a global solution of problem $(\mathcal{PB})$.

Local search in problem $(\mathcal{PB})$

Problem $(\mathcal{PB})$ as a general d.c. optimization problem

$$\left.\begin{array}{c}
F_0(x,y,z,\alpha,\beta;\mu) \stackrel{\triangle}{=} h_0(y,z;\mu) - g_0(x,y,z,\alpha,\beta;\mu) = \\
= f(x,y,z) - \mu p(y,z,\alpha,\beta) \uparrow \max\limits_{x,y,z,\alpha,\beta}, \quad (x,y,z) \in S, \\
F_i(x,z,\alpha) = g_i(x,z,\alpha) - h_i(x,z) \le 0, \quad i = 1,\dots,n_1; \\
F_j(x,y,\beta) = g_j(x,y,\beta) - h_j(x,y) \le 0, \; j = n_1+1,\dots,n_1+n_2.
\end{array}\right\} \qquad (\mathcal{PB})$$

Linearized problem (at the point (x^s, y^s, z^s))

$$\left.\begin{array}{r}
\langle \nabla h_0(y^s,z^s;\mu),(y^s,z^s)\rangle - g_0(x,y,z,\alpha,\beta;\mu) \uparrow \max\limits_{x,y,z,\alpha,\beta}, (x,y,z) \in S, \\
g_i(x,z,\alpha) - \langle \nabla h_i(x^s,z^s),(x,z)-(x^s,z^s)\rangle - h_i(x^s,z^s) \le 0, \\
i = 1,\dots,n_1; \\
g_j(x,y,\beta) - \langle \nabla h_j(x^s,y^s),(x,y)-(x^s,y^s)\rangle - h_j(x^s,y^s) \le 0, \\
j = n_1+1,\dots,n_1+n_2.
\end{array}\right\} \qquad (\mathcal{PL}_B)$$

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Thank you for your attention!