Nonlinear Fixed-Time Control Protocol for Uniform Allocation of Agents on a Segment

S. E. Parsegov

Institute of Control Sciences, Russian Academy of Sciences, Moscow

4th Traditional Youth Summer School “Control, Information, and Optimization”
Zvenigorod, 21/06/2012

Contents

Contents

Contents

Contents

Contents

Contents

Contents

Uniform Allocation on a Segment

Uniform Allocation on a Segment

Consider a group of n indexed mobile agents

Uniform Allocation on a Segment

Consider a group of n indexed mobile agents

— $x_i(t) \in \mathbb{R}$ is the position of each agent, $i = 1, \dots, n$

Uniform Allocation on a Segment

Consider a group of n indexed mobile agents

- $x_i(t) \in \mathbb{R}$ is the position of each agent, $i = 1, \ldots, n$
- x_0, x_{n+1} denote the fixed endpoints of the segment

Uniform Allocation on a Segment

Consider a group of n indexed mobile agents

- $x_i(t) \in \mathbb{R}$ is the position of each agent, $i = 1, \ldots, n$
- x_0, x_{n+1} denote the fixed endpoints of the segment

$$\dot{x}_i = u_i, \quad i = 1, \ldots, n, \tag{1}$$

$x = [x_1, x_2, \ldots, x_n]^\top$

Uniform Allocation on a Segment

Consider a group of n indexed mobile agents

— $x_i(t) \in \mathbb{R}$ is the position of each agent, $i = 1, \ldots, n$
— x_0, x_{n+1} denote the fixed endpoints of the segment

$$\dot{x}_i = u_i, \quad i = 1, \ldots, n, \tag{1}$$

$x = [x_1, x_2, \ldots, x_n]^\top$

The objective is to design a feedback control protocol that:

Uniform Allocation on a Segment

Consider a group of n indexed mobile agents

— $x_i(t) \in \mathbb{R}$ is the position of each agent, $i = 1, \dots, n$
— x_0, x_{n+1} denote the fixed endpoints of the segment

$$\dot{x}_i = u_i, \quad i = 1, \dots, n, \tag{1}$$

$x = [x_1, x_2, \dots, x_n]^\top$

The objective is to design a feedback control protocol that:

- guarantees the *equidistant allocation* of the agents on the segment in *fixed time for any initial conditions*

Uniform Allocation on a Segment

Consider a group of n indexed mobile agents

— $x_i(t) \in \mathbb{R}$ is the position of each agent, $i = 1, \ldots, n$
— x_0, x_{n+1} denote the fixed endpoints of the segment

$$\dot{x}_i = u_i, \quad i = 1, \ldots, n, \tag{1}$$

$x = [x_1, x_2, \ldots, x_n]^\top$

The objective is to design a feedback control protocol that:

- guarantees the *equidistant allocation* of the agents on the segment in *fixed time for any initial conditions*
- exploits the *information only about the distances* between each of the agents and its two indexed neighbors so that

$$u_i = u_i(x_{i-1} - x_i, \;\; x_{i+1} - x_i), \quad i = 1, \ldots, n \tag{2}$$

Uniform Allocation on a Segment

Uniform Allocation on a Segment

I. Linear approach (Darboux 1878; Wagner, Bruckstein 1997; Lin 2005; Petrikevich, Shcherbakov 2010; Parsegov 2011)

Uniform Allocation on a Segment

I. Linear approach (Darboux 1878; Wagner, Bruckstein 1997; Lin 2005; Petrikevich, Shcherbakov 2010; Parsegov 2011)

$$\dot{x}_i = u_i,$$

$$u_i = \frac{x_{i+1} + x_{i-1}}{2} - x_i, \quad i = 1, \ldots, n. \tag{3}$$

I. Linear approach (Darboux 1878; Wagner, Bruckstein 1997; Lin 2005; Petrikevich, Shcherbakov 2010; Parsegov 2011)

$$\dot{x}_i = u_i,$$

$$u_i = \frac{x_{i+1} + x_{i-1}}{2} - x_i, \quad i = 1, \ldots, n. \tag{3}$$

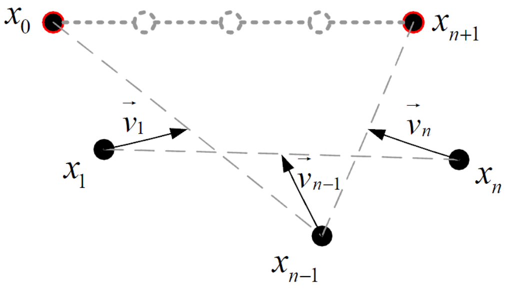

Fig.1 Idea of the control protocol

Uniform Allocation on a Segment

Uniform Allocation on a Segment

Overall dynamics

$$\dot{x} = Ax + b, \tag{4}$$

Uniform Allocation on a Segment

Overall dynamics

$$\dot{x} = Ax + b, \tag{4}$$

where the matrix A and the vector b are of the form

$$A = \begin{bmatrix} -1 & 0.5 & 0 & \dots & 0 \\ 0.5 & -1 & 0.5 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 0.5 & -1 \end{bmatrix} \in \mathbb{R}^{n \times n}, \tag{5}$$

$$b = [0.5x_0, 0, \dots, 0.5x_{n+1}]^\top \in \mathbb{R}^n. \tag{6}$$

Uniform Allocation on a Segment

Overall dynamics

$$\dot{x} = Ax + b, \tag{4}$$

where the matrix A and the vector b are of the form

$$A = \begin{bmatrix} -1 & 0.5 & 0 & \dots & 0 \\ 0.5 & -1 & 0.5 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 0.5 & -1 \end{bmatrix} \in \mathbb{R}^{n \times n}, \tag{5}$$

$$b = [0.5x_0, 0, \dots, 0.5x_{n+1}]^\top \in \mathbb{R}^n. \tag{6}$$

The control protocol (3) satisfies condition (2) but ...

Overall dynamics

$$\dot{x} = Ax + b, \tag{4}$$

where the matrix A and the vector b are of the form

$$A = \begin{bmatrix} -1 & 0.5 & 0 & \dots & 0 \\ 0.5 & -1 & 0.5 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 0.5 & -1 \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{5}$$

$$b = [0.5x_0, 0, \dots, 0.5x_{n+1}]^\top \in \mathbb{R}^n. \tag{6}$$

The control protocol (3) satisfies condition (2) but ...

Exponential convergence!

On Stability of an Equilibrium Point

On Stability of an Equilibrium Point

I. Exponential stability : $\|x(t, x_0)\| \leq Ce^{-at}, C, a > 0;$

On Stability of an Equilibrium Point

I. Exponential stability : $\|x(t, x_0)\| \leq Ce^{-at}, C, a > 0;$

$$\dot{x}(t) = -ax, \quad x(0) = x_0$$
$$x \in \mathbb{R}, t > 0$$

On Stability of an Equilibrium Point

I. Exponential stability : $\|x(t, x_0)\| \leq Ce^{-at}, C, a > 0$;

$$\dot{x}(t) = -ax, \quad x(0) = x_0$$
$$x \in \mathbb{R}, t > 0$$

II. Finite-time stability (Roxin 1966, Korobov 1979, Haimo 1986, Bhat & Bernstein 2000) : $\forall x_0 \in \mathbb{R}^n \; \exists T = T(x_0) : x(t, x_0) = 0, \forall t \geq T$;

On Stability of an Equilibrium Point

I. Exponential stability : $\|x(t, x_0)\| \leq Ce^{-at}, C, a > 0$;

$$\dot{x}(t) = -ax, \quad x(0) = x_0$$
$$x \in \mathbb{R}, t > 0$$

II. Finite-time stability (Roxin 1966, Korobov 1979, Haimo 1986, Bhat & Bernstein 2000) : $\forall x_0 \in \mathbb{R}^n \; \exists T = T(x_0) : x(t, x_0) = 0, \forall t \geq T$;

$$\dot{x}(t) = -x^{1/3}, \quad x(0) = x_0$$
$$x \in \mathbb{R}, t > 0, T(x_0) = \frac{3}{2} x_0^{2/3}$$

On Stability of an Equilibrium Point

I. Exponential stability : $\|x(t,x_0)\| \leq Ce^{-at}, C, a > 0$;

$$\dot{x}(t) = -ax, \quad x(0) = x_0$$
$$x \in \mathbb{R}, t > 0$$

II. Finite-time stability (Roxin 1966, Korobov 1979, Haimo 1986, Bhat & Bernstein 2000) : $\forall x_0 \in \mathbb{R}^n \; \exists T = T(x_0) : x(t,x_0) = 0, \forall t \geq T$;

$$\dot{x}(t) = -x^{1/3}, \quad x(0) = x_0$$
$$x \in \mathbb{R}, t > 0, T(x_0) = \frac{3}{2}x_0^{2/3}$$

III. Fixed-time stability (Polyakov 2011) $\exists T_{max} > 0 : x(t,x_0)=0, \; \forall t > T_{max}$, $\forall x_0 \in \mathbb{R}^n$;

On Stability of an Equilibrium Point

I. Exponential stability : $\|x(t,x_0)\| \leq Ce^{-at}, C, a > 0;$

$$\dot{x}(t) = -ax, \quad x(0) = x_0$$
$$x \in \mathbb{R}, t > 0$$

II. Finite-time stability (Roxin 1966, Korobov 1979, Haimo 1986, Bhat & Bernstein 2000) : $\forall x_0 \in \mathbb{R}^n \; \exists T = T(x_0) : x(t,x_0) = 0, \forall t \geq T;$

$$\dot{x}(t) = -x^{1/3}, \quad x(0) = x_0$$
$$x \in \mathbb{R}, t > 0, T(x_0) = \frac{3}{2}x_0^{2/3}$$

III. Fixed-time stability (Polyakov 2011) $\exists T_{\max} > 0 : x(t,x_0){=}0, \; \forall t > T_{\max},$ $\forall x_0 \in \mathbb{R}^n;$

$$\dot{x}(t) = -x^{1/3} - x^3, \quad x(0) = x_0$$
$$x \in \mathbb{R}, t > 0, T_{\max} = 2\frac{2}{3}$$

Fixed-Time Stability: Elements of the Theory

$$\dot{x} = g(t, x), \quad x(0) = x_0, \tag{7}$$

where $x \in \mathbb{R}^n$ and $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$. Assume that the origin is an equilibrium point of system (7).

Fixed-Time Stability: Elements of the Theory

$$\dot{x} = g(t,x), \quad x(0) = x_0, \tag{7}$$

where $x \in \mathbb{R}^n$ and $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$. Assume that the origin is an equilibrium point of system (7).

Definition (Polyakov 2011)

The origin is said to be a *fixed-time stable* equilibrium point of system (7) if it is globally finite-time stable and the settling-time function $T(x_0)$ is bounded, i.e., there exists $T_{\max} > 0$: $T(x_0) \leq T_{\max}\ \forall x_0 \in \mathbb{R}^n$.

Fixed-Time Stability: Elements of the Theory

$$\dot{x} = g(t,x), \quad x(0) = x_0, \tag{7}$$

where $x \in \mathbb{R}^n$ and $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$. Assume that the origin is an equilibrium point of system (7).

Definition (Polyakov 2011)

The origin is said to be a *fixed-time stable* equilibrium point of system (7) if it is globally finite-time stable and the settling-time function $T(x_0)$ is bounded, i.e., there exists $T_{\max} > 0$: $T(x_0) \le T_{\max}\ \forall x_0 \in \mathbb{R}^n$.

Lemma (Polyakov 2011)

If there exists a continuous radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ such that 1) $V(x) = 0 \Leftrightarrow x = 0$; 2) any solution $x(t)$ of (7) satisfies the inequality $D^* V(x(t)) \le -\alpha V^p(x(t)) + \beta V^q(x(t))$ for some $\alpha, \beta, p, q > 0$: $p < 1$, $q > 1$, then the origin is globally fixed-time stable for system (7) and the following estimate holds:

$$T(x_0) \le \frac{1}{\alpha(1-p)} + \frac{1}{\beta(q-1)} \quad \forall x_0 \in \mathbb{R}^n.$$

Fixed-Time Stability: New Result

Fixed-Time Stability: New Result

Refinement of Lemma 1

Refinement of Lemma 1

Let the constants p and q be of the form $p = 1 - \frac{1}{2\mu}$ and $q = 1 + \frac{1}{2\mu}$, $\mu > 1$.

Fixed-Time Stability: New Result

Refinement of Lemma 1

Let the constants p and q be of the form $p = 1 - \frac{1}{2\mu}$ and $q = 1 + \frac{1}{2\mu}$, $\mu > 1$.

Lemma (Parsegov, Polyakov, Shcherbakov 2012)

If there exists a continuous radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ such that 1) $V(x) = 0 \Leftrightarrow x = 0$; 2) any solution $x(t)$ of (7) satisfies the inequality $D^* V(x(t)) \leq -\alpha V^p(x(t)) - \beta V^q(x(t))$ for some $\alpha, \beta > 0$, $p = 1 - \frac{1}{2\mu}$, $q = 1 + \frac{1}{2\mu}$, $\mu > 1$, then the origin is globally fixed-time stable for system (7) and the following estimate holds:

$$T_{\max} := \frac{\pi\mu}{\sqrt{\alpha\beta}}, \forall x_0 \in \mathbb{R}^n.$$

Fixed-Time Stability: New Result

Refinement of Lemma 1

Let the constants p and q be of the form $p = 1 - \frac{1}{2\mu}$ and $q = 1 + \frac{1}{2\mu}$, $\mu > 1$.

Lemma (Parsegov, Polyakov, Shcherbakov 2012)

If there exists a continuous radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ *such that 1)* $V(x) = 0 \Leftrightarrow x = 0$*; 2) any solution* $x(t)$ *of (7) satisfies the inequality* $D^* V(x(t)) \leq -\alpha V^p(x(t)) - \beta V^q(x(t))$ *for some* $\alpha, \beta > 0$, $p = 1 - \frac{1}{2\mu}$, $q = 1 + \frac{1}{2\mu}$, $\mu > 1$, *then the origin is globally fixed-time stable for system (7) and the following estimate holds:*

$$T_{\max} := \frac{\pi\mu}{\sqrt{\alpha\beta}}, \forall x_0 \in \mathbb{R}^n.$$

Less conservative estimate of the settling time!

Uniform Allocation on a Segment: New Result

Uniform Allocation on a Segment: New Result

II. Nonlinear approach

Uniform Allocation on a Segment: New Result

II. Nonlinear approach

Each agent has single-integrator dynamics

$$\dot{x}_i = u_i,$$

Uniform Allocation on a Segment: New Result

II. Nonlinear approach

Each agent has single-integrator dynamics

$$\dot{x}_i = u_i,$$

Control protocol

$$u_i = \phi\left(\frac{x_{i+1} + x_{i-1}}{2} - x_i\right). \tag{8}$$

Uniform Allocation on a Segment: New Result

II. Nonlinear approach

Each agent has single-integrator dynamics

$$\dot{x}_i = u_i,$$

Control protocol

$$u_i = \phi\left(\frac{x_{i+1} + x_{i-1}}{2} - x_i\right). \tag{8}$$

Function $\phi(s), s \in \mathbb{R}$

$$\phi(s) := \alpha s^{[p]} + \beta s^{[q]}, \qquad 0 < p < 1, \quad q > 1, \quad \alpha, \beta > 0, \tag{9}$$

$$s^{[k]} := \operatorname{sign}(s)|s|^k. \tag{10}$$

II. Nonlinear approach

Each agent has single-integrator dynamics

$$\dot{x}_i = u_i,$$

Control protocol

$$u_i = \phi\left(\frac{x_{i+1} + x_{i-1}}{2} - x_i\right). \tag{8}$$

Function $\phi(s), s \in \mathbb{R}$

$$\phi(s) := \alpha s^{[p]} + \beta s^{[q]}, \qquad 0 < p < 1, \quad q > 1, \quad \alpha, \beta > 0, \tag{9}$$

$$s^{[k]} := \text{sign}(s)|s|^k. \tag{10}$$

Overall dynamics

$$\dot{x} = \bar{\phi}(Ax + b), \tag{11}$$

$$\bar{\phi}(z) := [\phi(z_1), \phi(z_2), \ldots, \phi(z_n)]^\top,$$
$$z = [z_1, z_2, \ldots, z_n]^\top \in \mathbb{R}^n.$$

Stability and Convergence Time Analysis

Stability and Convergence Time Analysis

1) Using the results of Lemma 1

Stability and Convergence Time Analysis

1) Using the results of Lemma 1

Theorem (Parsegov, Polyakov, Shcherbakov 2012)

Let the control protocol u_i be defined by (8) *with* $\alpha > 0$, $\beta > 0$, $0 < p < 1$, $q > 1$. *Then the agents of the multi-agent system* (1) *allocate equidistantly on the segment in a fixed time which is globally (over all initial conditions) bounded by* $T_{\max}$ *given by*

$$T_{\max} := \frac{2}{\alpha(1-p)(2|\hat{\lambda}|)^{\frac{p+1}{2}}} + \frac{2n^{\frac{q-1}{2}}}{\beta(q-1)(2|\hat{\lambda}|)^{\frac{q+1}{2}}}, \qquad (12)$$

where $\hat{\lambda} = \max_k \lambda_k(A)$.

Stability and Convergence Time Analysis

1) Using the results of Lemma 1

Theorem (Parsegov, Polyakov, Shcherbakov 2012)

Let the control protocol u_i be defined by (8) with $\alpha > 0$, $\beta > 0$, $0 < p < 1$, $q > 1$. Then the agents of the multi-agent system (1) allocate equidistantly on the segment in a fixed time which is globally (over all initial conditions) bounded by $T_{\max}$ given by

$$T_{\max} := \frac{2}{\alpha(1-p)(2|\hat{\lambda}|)^{\frac{p+1}{2}}} + \frac{2n^{\frac{q-1}{2}}}{\beta(q-1)(2|\hat{\lambda}|)^{\frac{q+1}{2}}}, \tag{12}$$

where $\hat{\lambda} = \max_k \lambda_k(A)$.

2) Special choice of parameters p and q for using Lemma 2

Stability and Convergence Time Analysis

1) Using the results of Lemma 1

Theorem (Parsegov, Polyakov, Shcherbakov 2012)

Let the control protocol u_i be defined by (8) with $\alpha > 0$, $\beta > 0$, $0 < p < 1$, $q > 1$. Then the agents of the multi-agent system (1) allocate equidistantly on the segment in a fixed time which is globally (over all initial conditions) bounded by $T_{\max}$ given by

$$T_{\max} := \frac{2}{\alpha(1-p)(2|\hat{\lambda}|)^{\frac{p+1}{2}}} + \frac{2n^{\frac{q-1}{2}}}{\beta(q-1)(2|\hat{\lambda}|)^{\frac{q+1}{2}}}, \tag{12}$$

where $\hat{\lambda} = \max_k \lambda_k(A)$.

2) Special choice of parameters p and q for using Lemma 2

Theorem (Parsegov, Polyakov, Shcherbakov 2012)

If, under the conditions of Theorem 1, the constants p and q of system (11) are chosen as $p = 1 - \frac{1}{\mu}$ and $q = 1 + \frac{1}{\mu}$, $\mu > 1$, then the settling time estimate can be found as

$$T_{\max} := \frac{\pi \mu n^{\frac{1}{4\mu}}}{2|\hat{\lambda}|\sqrt{\alpha\beta}}. \tag{13}$$

Numerical Example

Numerical Example

System parameters: $n = 3, \quad x_0 = 0, \quad x_{n+1} = 16$
Initial conditions: $x_1(0) = -5, \quad x_2(0) = -5\sqrt{2}, \quad x_3(0) = -5$

Numerical Example

System parameters: $n = 3, \quad x_0 = 0, \quad x_{n+1} = 16$
Initial conditions: $x_1(0) = -5, \quad x_2(0) = -5\sqrt{2}, \quad x_3(0) = -5$

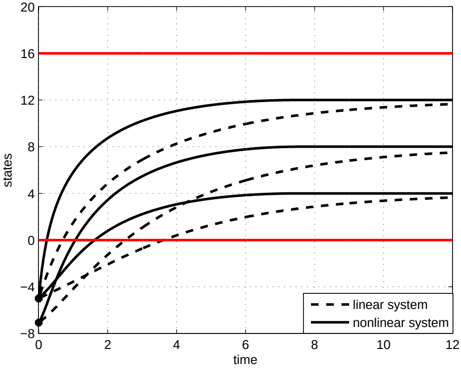

Fig.2 Trajectories of the system under linear and nonlinear protocols

Conclusions

Conclusions

1. nonlinear control protocol for finite-time equidistant allocation of agents is developed

Conclusions

1. nonlinear control protocol for finite-time equidistant allocation of agents is developed
2. the estimates (12), (13) yield an extremely important conclusion: the system can be forced to have any *a priori* specified settling time by properly choosing the parameters μ, α, and β regardless of the initial positions of the agents (fixed-time convergence)

Conclusions

1. nonlinear control protocol for finite-time equidistant allocation of agents is developed
2. the estimates (12), (13) yield an extremely important conclusion: the system can be forced to have any *a priori* specified settling time by properly choosing the parameters μ, α, and β regardless of the initial positions of the agents (fixed-time convergence)
3. Theorem 1 presents quite a conservative settling time estimate (12), since its proof is based on the results of Lemma 1. A more accurate estimate (13) is presented in Theorem 2

References

S.P. Bhat, D.S. Bernstein,
Finite-time stability of continuous autonomous systems,
SIAM Journal of Control and Optimization, 38(3):751–766, 2000.

J.G. Darboux,
Sur un problème de géométrie élémentaire,
Bulletin des Sciences Mathématiques et Astronomiques, 2(1): 298–304, 1878.

V.T. Haimo,
Finite time controllers,
SIAM Journal of Control and Optimization, 24(4):760–770, 1986.

Z. Lin, B. Francis, M. Maggiore,
Necessary and sufficient graphical conditions for formation control of unicycles,
IEEE Transactions on Automatic Control, 50(1):121–127, 2005.

S. E. Parsegov,
Allocation of agents on a line: simple algorithm and generalizations,
Proc. 14th Baltic olympiad on automatic control, Saint-Petersburg, Russia, Sep. 21–23, 2011, pp. 119–125.

References

A. Polyakov,
Nonlinear feedback design for fixed-time stabilization of linear control systems,
IEEE Transactions on Automatic Control, 2011, doi: 10.1109/TAC.2011.2179869.

A. Polyakov,
Fixed-time stabilization of linear systems via sliding mode control,
Proc. 12th IEEE International Workshop on Variable Structure Systems, Mumbai, India, Jan. 12–14, 2012, pp. 1–6.

E. Roxin,
On finite stability in control cystems,
Rendiconti del Circolo Matematico di Palermo, 15(3):273–283, 1966.

P. Shcherbakov,
Multiagent systems: From the Van Loan scheme to simple linear algorithms,
Proc. PhysCon 2011, Léon, Spain, Sep. 5–8, 2011.

I.A. Wagner, A.M. Bruckstein,
Row straightening via local interactions,
Circuits, Systems, and Signal Processing, 16(3): 287–305, 1997.

Thank you for attention!