Stabilizing nonminimum-phase systems

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Stabilization of $x_0 = 0$

$$\dot{x} = A(x) + \sum_{i=1}^{m} B_i(x) u_i, \tag{1}$$

$$\begin{aligned} & x \in R^n, u = (u_1, \ldots, u_m)^T \in R^m, \\ & A(x) = (a_1(x), \ldots, a_n(x))^T, \\ & A(0) = 0, \end{aligned} \tag{2}$$

$$\begin{aligned} & B(x) = (B_1(x), \ldots, B_m(x)), \\ & B_j(x) = (b_j^1(x), \ldots, b_j^n(x))^T, \\ & j = \overline{1, m} \\ & A(x), B_j(x) \in C^\infty(x) \end{aligned} \tag{3}$$

Relative Degree

$$\dot{x} = A(x) + B(x)u, \tag{4}$$

$$y = h(x)$$

Def.: ρ — relative degree in x_0

$$\Updownarrow$$

$$L_B L_A^i h(x) = 0,\ i = \overline{0, \rho - 2},$$

$$L_B L_A^{\rho-1} h(x^0) \neq 0.$$

Normal Form

$$\begin{aligned} &\dot{z}_1 = z_2, \ldots, \dot{z}_{r-1} = z_r, \\ &\dot{z}_r = f(z,\eta) + g(z,\eta)u, \\ &\dot{\eta} = q(z,\eta), \\ &y = z_1, \\ &g(0,0) \neq 0. \end{aligned} \tag{5}$$

$$L_B \eta = 0$$

Zero dynamics:

$$\dot{\eta} = q(0,\eta),$$

Minimum-phase system — stabilizing all variables.
Partial feedback linearization:

$$u = \left(-L_A^{\rho} h(x) - \sum_{k=0}^{\rho-1} c_k L_A^k h(x) \right) \Big/ L_B L_A^{\rho-1} h(x) \,, \tag{6}$$

Nonminimum-phase system — ?

- Case $\rho = 1$ и $\rho = 2$ — Крищенко, Ткачев, Панфилов (2010)
  Method for finding minimum-phase output.
- Case $\rho > 2$ studied partially.
- Case of multiple inputs is studied partially.

$\rho > 2, \mathrm{q}(\mathrm{z}, \eta) = \mathrm{p}(\mathrm{z}_1, \eta)$ I

Theorem 1 (Ткачев)

Let normal form of (4) with output $\mathrm{y} = \mathrm{h}(\mathrm{x})$ in the neighborhood of $\mathrm{x} = 0$ be (5), and $\mathrm{q}(\mathrm{z}, \eta) \equiv \mathrm{p}(\mathrm{y}, \eta) \equiv \mathrm{p}(\mathrm{z}_1, \eta)$. For (4) to have an output with relative degree $\rho = \mathrm{r}$ in $\mathrm{x} = 0$ and asymptotically stable zero dynamics, it is necessary and sufficient, for the equilibrium $\eta = 0$ of

$$\dot{\eta} = \mathrm{p}(\mathrm{v}, \eta) \tag{7}$$

with input v to be stabilizable by smooth feedback $\mathrm{v} = \mathrm{v}(\eta)$.

$\rho > 2, q(z, \eta) = p(z_1, \eta)$ II

Theorem 1 (Ткачев)

To each such stabilizing feedback corresponds an output $y = z_1 - v(\eta) = h(x) - v(\Psi(x))$ of (4) with relative degree $\rho = r$ in $x = 0$ and asymptotically stable zero dynamics.

$\rho > 2, q(z,\eta) = p(z_1, z_2, \eta)$

Theorem 2 (Ткачев)

If in (5) $q(z,\eta) \equiv p(z_1, z_2, \eta)$, and feedback $v_1 = v_1(\eta)$, $v_2 = v_2(\eta)$, $v_1(0) = 0$, $v_2(0) = 0$, stabilizes the equilibrium $\eta = 0$ of

$$\dot{\eta} = p(v_1, v_2, \eta)$$

and fulfills conditions

$$\left.\frac{dv_1(\eta)}{dt}\right|_{\dot{\eta}=p(v_1(\eta),v_2(\eta),\eta)} = v_2(\eta), \tag{8}$$

$$1 - v_1^{'} p_{z_2}^{'}(0,0,0) \neq 0,$$

then (5) with output $\phi(z,\eta) = z_1 - v_1(\eta)$ has relative degree r in $(z,\eta) = 0$, and zero dynamics, corresponding to this output is asymptotically stable in $\eta = 0$.

Example 1

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{m}(-k_s x_1 - b_d x_2 + \Lambda_a x_3) \\
\dot{x}_3 &= -\alpha x_2 - \beta x_3 + (\gamma\sqrt{Ps - x_3})x_4 \\
\dot{x}_4 &= -\frac{1}{\tau}x_4 + \frac{K_a}{\tau}u.
\end{aligned}
$$

$$z_1 = x_2$$

Example 1 (continued).

$$\rho = 3$$

Dimension of η subsystem

$$\mathrm{n} - \rho = 1.$$

$\eta = \mathrm{x}_1$

$$\dot{\eta} = \mathrm{x}_2 = \mathrm{z}_1.$$

Zero dynamics

$$\dot{\eta} = \mathrm{p}(0, \eta) = 0.$$

$$\dot{\eta} = \mathrm{v}(\eta)$$

Let $\mathrm{v}(\eta) = -\eta$.

Example 1 (continued).

By theorem (1) there exists an output

$$\mathrm{y} = \overline{\mathrm{z}_1} = \mathrm{z}_1 - \mathrm{v}(\eta) = \mathrm{x}_2 + \mathrm{x}_1$$

with $\rho = 3$ and asymptotically stable zero dynamics. Indeed,

$$\eta = \mathrm{x}_1, \dot{\overline{\eta}} = \mathrm{x}_2 = \overline{\mathrm{z}_1} - \mathrm{x}_1 = \overline{\mathrm{z}_1} - \overline{\eta},$$

$$\dot{\overline{\eta}}|_{\overline{\mathrm{z}}=0} = -\overline{\eta}$$

Example 5. Inverted pendulum

$$\ddot{x} = \frac{mg\cos\theta\sin\theta - ml\sin\theta\dot{\theta}^2 + f}{M + m\sin^2(\theta)}$$
$$\ddot{\theta} = \frac{(M+m)g\sin\theta - ml\cos\theta\sin\theta\dot{\theta}^2 + f\cos\theta}{(M + m\sin^2\theta)l}$$

$$y = \theta$$

$$z_1 = \theta, z_2 = \dot{\theta}.$$

$$|\theta| < \frac{\pi}{2}, \rho = 2$$

$$B(x) = \frac{1}{M + m\sin^2\theta}\frac{\partial}{\partial\dot{x}} + \frac{\cos\theta}{(M + m\sin^2\theta)l}\frac{\partial}{\partial\dot{\theta}}$$

$$\eta_1 = x, \eta_2 = l\dot{\theta} - \dot{x}\cos\theta$$

Example 5 (continued)

$$z_1 = \theta,\ z_2 = \dot{\theta},\ \eta_1 = x,\ \eta_2 = l\dot{\theta} - \dot{x}\cos\theta$$

$$\dot{\eta}_1 = \frac{lz_2 - \eta_2}{\cos z_1},$$

$$\dot{\eta}_2 = \left(g + \frac{lz_2 - \eta_2}{\cos z_1} z_2\right) \sin z_1.$$

$$\begin{aligned} \dot{\eta}_1 &= -\eta_2, \\ \dot{\eta}_2 &= 0. \end{aligned}$$

Example 5 (continued)

$$\dot{\eta}_1 = \frac{\mathrm{l}\mathrm{v}_2 - \eta_2}{\cos \mathrm{v}_1},$$
$$\dot{\eta}_2 = \left(\mathrm{g} + \frac{\mathrm{l}\mathrm{v}_2 - \eta_2}{\cos \mathrm{v}_1}\mathrm{z}_2\right)\sin \mathrm{v}_1,$$

v_1 and v_2 fulfill the condition

$$\frac{\partial \mathrm{v}_1}{\partial \eta_1}\frac{\mathrm{l}\mathrm{v}_2 - \eta_2}{\cos \mathrm{v}_1} + \frac{\partial \mathrm{v}_1}{\partial \eta_2}\left(\mathrm{g} + \frac{\mathrm{l}\mathrm{v}_2 - \eta_2}{\cos \mathrm{v}_1}\mathrm{v}_2\right)\sin \mathrm{v}_1 = \mathrm{v}_2.$$

Example 5 (continued)

Linearizing zero dynamics

$$\dot{\eta}_1 = -\eta_2 + \mathrm{l}\mathrm{v}_2,$$
$$\dot{\eta}_2 = \mathrm{g}\mathrm{v}_1.$$

$$\overline{\mathrm{y}} = \theta + \mathrm{k}_1\eta_1 + \mathrm{k}_2\eta_2 = \theta + \mathrm{k}_1\mathrm{x} + \mathrm{k}_2(\mathrm{l}\dot{\theta} - \dot{\mathrm{x}}\cos\theta).$$

Either $\mathrm{k}_2 > 0$ and $-\frac{1}{\mathrm{l}} < \mathrm{k}_1 < 0$ or $\mathrm{k}_1 < -\frac{1}{\mathrm{l}}$ and $\mathrm{k}_2 < 0$.

Example 5 (continued)

$$\dot{\bar{z}}_1 = \bar{z}_2,$$

$$\dot{\bar{z}}_2 = \left(\frac{k_1(lz_2-\eta_2)\sin z_1}{\cos^2 z_1} + \frac{k_2(lz_2-\eta_2)z_2\sin^2 z_1}{\cos^2 z_1}\right) z_2 +$$

$$+ k_2(g\cos z_1 + (lz_2 - \eta_2)z_2)z_2 +$$

$$+ \left(1 + \frac{k_1 l}{\cos z_1} + k_2\left(\frac{2lz_2-\eta_2}{\cos z_1}\right)\sin z_1\right) \times$$

$$\times \frac{(M+m)\sin z_1 g - mlz_2^2\cos z_1 \sin z_1}{(M+m\sin^2 z_1)l} -$$

$$- \left(\frac{k_1}{\cos z_1} + \frac{k_2 z_2 \sin z_1}{\cos z_1}\right)\left(g + \frac{(lz_2-\eta_2)z_2}{\cos z_1}\right)\sin z_1 +$$

$$+ \frac{\cos z_1 + k_1 l + k_2(2lz_2-\eta_2)\sin z_1}{(M+m\sin^2 z_1)} u = \bar{f} + \bar{g}u,$$

$\bar{z}_1 = \bar{y}$, $\bar{z}_2 = z_2 + k_1\frac{lz_2-\eta_2}{\cos z_1} + k_2\left(g + \frac{lz_2-\eta_2}{\cos z_1}z_2\right)\sin z_1.$

Stabilizing feedback

Necessary condition, $\rho > 2, q(z, \eta) = p(z_1, z_2, \eta)$

Theorem 3

If in (5) $q(z, \eta) \equiv p(z_1, z_2, \eta)$, and there exists an output $\phi(z_1, \eta)$, such that its relative degree is r in $x = 0$, zero dynamics is asymptotically stable, $\frac{\partial \phi(z_1,\eta)}{\partial z_1}(0, 0) \neq 0$. Then there exist functions $v_1 = v_1(\eta)$, $v_2 = v_2(\eta)$, such that $v_1(0) = 0$, $v_2(0) = 0$, $v_1 = v_1(\eta)$, $v_2 = v_2(\eta)$ stabilize the equilibrium $\eta = 0$ of

$$\dot{\eta} = p(v_1, v_2, \eta)$$

and fulfill conditions

$$\left.\frac{dv_1(\eta)}{dt}\right|_{\dot{\eta}=p(v_1(\eta),v_2(\eta),\eta)} = v_2(\eta). \tag{9}$$

Multiple inputs

$$\dot{x} = A(x) + \sum_{i=1}^{m} B_i(x)u_i, \tag{10}$$

$$\begin{aligned}
&\dot{z}_1^1 = z_2^1, \dots, \dot{z}_{r_1-1}^i = z_{r_1}^1,\\
&\dot{z}_{r_1}^1 = f_1(z,\eta) + g_{11}(z,\eta)u_1 + \dots + g_{1m}(z,\eta)u_m,\\
&\dots,\\
&\dot{z}_1^m = z_2^m, \dots, \dot{z}_{r_m-1}^m = z_{r_m}^m,\\
&\dot{z}_{r_m}^m = f_m(z,\eta) + g_{m1}(z,\eta)u_1 + \dots + g_{mm}(z,\eta)u_m,\\
&\dot{\eta} = q(z,\eta),\\
&y = (z_1^1, z_1^2, \dots, z_1^m)^T,
\end{aligned} \tag{11}$$

matrix $(g_{ij}(0,0))_{i,j=\overline{1,m}}$ is nondegenerate.
$\mathrm{span}(B_1, \dots, B_m)$ — involutive and regular.

$$\rho = (\mathrm{r}, \ldots, \mathrm{r})$$

— homogenous relative degree
Cases $\rho = (1, \ldots, 1)$ and $\rho = (2, \ldots, 2)$ were studied by Ткачев(2010).
Case $\rho = (\mathrm{r}, \ldots, \mathrm{r}), \mathrm{r} > 2$ not yet studied.

$\rho = (\mathrm{r}, \ldots, \mathrm{r}), \mathrm{q}(\mathrm{z}, \eta) = \mathrm{p}(\mathrm{z}_1^1, \ldots, \mathrm{z}_1^{\mathrm{m}}, \eta)$

Theorem 4

Let normal form of (1) with output $\mathrm{y} = (\mathrm{h}_1(\mathrm{x}), \ldots, \mathrm{h}_{\mathrm{m}}(\mathrm{x}))$ in the neighborhood of $\mathrm{x} = 0$ be (11), and $\mathrm{q}(\mathrm{z}, \eta) = \mathrm{p}(\mathrm{z}_1^1, \ldots, \mathrm{z}_1^{\mathrm{m}}, \eta)$. For (11) to have an output with relative degree $\rho = (\mathrm{l}, \ldots, \mathrm{l})$ in $\mathrm{x} = 0$ and asymptotically stable zero dynamics, it is necessary and sufficient for equilibrium point $\eta = 0$ of nonlinear system

$$\dot{\eta} = \mathrm{p}(\mathrm{v}_1, \ldots, \mathrm{v}_{\mathrm{m}}, \eta) \tag{12}$$

with controls $\mathrm{v}_1, \ldots, \mathrm{v}_{\mathrm{m}}$ to be stabilizable with smooth feedback $\mathrm{v}_1 = \mathrm{v}_1(\eta), \ldots, \mathrm{v}_{\mathrm{m}} = \mathrm{v}_{\mathrm{m}}(\eta)$.

To each such stabilizing input in (12) corresponds an output

$$y = (z_1^1 - v_1(\eta), \ldots, z_1^m - v_m(\eta)) = (h_1(x) - v_1(\Psi(x)), \ldots, h_m(x) - v_m(\Psi(x)))$$

of (11) with relative degree $\rho = (1, \ldots, 1)$ in $x = 0$ and asymptotically stable zero dynamics.

Example 3

$$
\begin{aligned}
&\dot{z}_1^1 = z_2^1, \ldots, \dot{z}_{l-1}^i = z_l^1, \\
&\dot{z}_l^1 = u_1, \\
&\dot{z}_1^2 = z_2^2, \ldots, \dot{z}_{l-1}^2 = z_l^2, \\
&\dot{z}_l^2 = u_2, \\
&\dot{\eta} = z_1^1 + z_1^2 + \eta,
\end{aligned} \tag{13}
$$

$$\dot{\eta} = p(0, \eta) = \eta.$$

$\eta = 0$ unstable.

$$\dot{\eta} = p(v_1, \ldots, v_m, \eta)$$

$$v_1 = -\eta, v_2 = -\eta$$

Example 3 (continued).

$$\overline{z_1^1} = \phi_1 = z_1^1 + \eta,$$

$$\overline{z_1^2} = \phi_2 = z_1^2 + \eta,$$

$$\bar{\eta} = \eta,$$

$$\dot{\bar{\eta}} = \bar{\eta} + z_1^1 + z_1^2 = -\bar{\eta} + \overline{z_1^1} + \overline{z_1^2},$$

$$\dot{\bar{\eta}} = p(0, \bar{\eta}) = -\bar{\eta}.$$

$\bar{\eta} = 0$ stable.

$\rho = (\mathrm{r}, \ldots, \mathrm{r}), \mathrm{q}(\mathrm{z}, \eta) = \mathrm{p}(\mathrm{z}_1^1, \ldots, \mathrm{z}_1^\mathrm{m}, \mathrm{z}_2^1, \ldots, \mathrm{z}_2^\mathrm{m}, \eta)$ I

Theorem 5

If in (11) $\mathrm{q}(\mathrm{z}, \eta) \equiv \mathrm{p}(\mathrm{z}_1^1, \ldots, \mathrm{z}_1^\mathrm{m}, \mathrm{z}_2^1, \ldots, \mathrm{z}_2^\mathrm{m}, \eta)$, $\rho = (\mathrm{l}, \ldots \mathrm{l})$, and controls $\mathrm{v}_1^1 = \mathrm{v}_1^1(\eta), \ldots, \mathrm{v}_1^\mathrm{m} = \mathrm{v}_1^\mathrm{m}(\eta), \mathrm{v}_2^1 = \mathrm{v}_2^1(\eta), \ldots, \mathrm{v}_2^\mathrm{m} = \mathrm{v}_2^\mathrm{m}(\eta)$, $\mathrm{v}_1^1(0) = 0, \ldots, \mathrm{v}_1^\mathrm{m}(0) = 0, \mathrm{v}_2^1(0) = 0, \ldots, \mathrm{v}_2^\mathrm{m}(0) = 0$ stabilize the equilibrium $\eta = 0$ of

$$\dot{\eta} = \mathrm{p}(\mathrm{v}_1^1, \ldots, \mathrm{v}_1^\mathrm{m}, \mathrm{v}_2^1, \ldots, \mathrm{v}_2^\mathrm{m}, \eta),$$

and fulfill conditions

$$\left.\frac{\mathrm{d}\mathrm{v}_1^\mathrm{i}(\eta)}{\mathrm{d}\mathrm{t}}\right|_{\dot{\eta}=\mathrm{p}(\mathrm{v}_1^1,\ldots,\mathrm{v}_1^\mathrm{m},\mathrm{v}_2^1,\ldots,\mathrm{v}_2^\mathrm{m},\eta)} = \mathrm{v}_2^\mathrm{i}(\eta), \tag{14}$$

$$\det(\mathrm{E} - (\mathrm{v}_1)'_\eta \mathrm{p}'_{\mathrm{z}_2^1,\ldots,\mathrm{z}_2^\mathrm{m}}(0,0,0)) \neq 0,$$

$\rho = (r, \ldots, r), q(z, \eta) = p(z_1^1, \ldots, z_1^m, z_2^1, \ldots, z_2^m, \eta)$ II

Theorem 5

then (11) with output $\phi(z, \eta) = (\phi_1(z, \eta), \ldots, \phi_m(z, \eta))$, $\phi_i(z, \eta) = z_1^i - v_1^i(\eta)$ has relative degree $\rho = (l, \ldots, l)$ in $(z, \eta) = 0$, and zero dynamics, corresponding to this output, is asymptotically stable in $\eta = 0$.

Example 4

$$\mathrm{m} = 2, \mathrm{n} - \mathrm{r} = 4$$

$$\dot{\eta} = \mathrm{A}\eta + \mathrm{B}^1\mathrm{v}^1 + \mathrm{B}^2\mathrm{v}^2\,, \tag{15}$$

$$\mathrm{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -10 & 0 \end{bmatrix}, \qquad \mathrm{B}^1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathrm{B}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}.$$

$$\mathrm{v}^1 = -\mathrm{K}\eta$$

$$\left.\frac{d\mathrm{v}_1^{\mathrm{i}}(\eta)}{dt}\right|_{\dot{\eta}=\mathrm{p}(\mathrm{v}_1^1,\ldots,\mathrm{v}_1^{\mathrm{m}},\mathrm{v}_2^1,\ldots,\mathrm{v}_2^{\mathrm{m}},\eta)} = \mathrm{v}_2^{\mathrm{i}}(\eta),$$

$$\mathrm{v}^2 = (\mathrm{E} + \mathrm{K}\,\mathrm{B}^2)^{-1}\mathrm{K}(\mathrm{B}^1\mathrm{K} - \mathrm{A})\,\eta$$

Example 4 (continued).

$$\dot{\eta} = A\eta + B^1 v^1 + B^2 v^2 = (A - B^1 K + B^2 (E + K B^2)^{-1} (-KA + K B^1 K))\eta = R\,\eta.$$

$$K = \begin{bmatrix} K_{11} & 0 & K_{13} & 0 \\ K_{21} & K_{22} & K_{23} & K_{24} \end{bmatrix},$$

$K_{11} > 0$, $K_{22} > 0$, $K_{13} < 0$, arbitrary K_{21} , K_{23} , K_{24} .

$$K_{21}\,,\ K_{23}\,,\ K_{24} = 0$$

$$\bar{z}^1 = z^1 + K\eta, \bar{z}^2 = \dot{\bar{z}}^1, \bar{\eta} = \eta$$

$$\dot{\bar{\eta}} = R\bar{\eta}$$

Results

Affine dynamical system

- Normal form
  - Minimum-phase
    - Linearizing feedback
  - Nonminimum-phase
    - Virtual output method
    - New outputs
    - Minimum-phase system
    - Linearizing feedback

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