The “Butterfly” Robot
Virtual Constraint Approach

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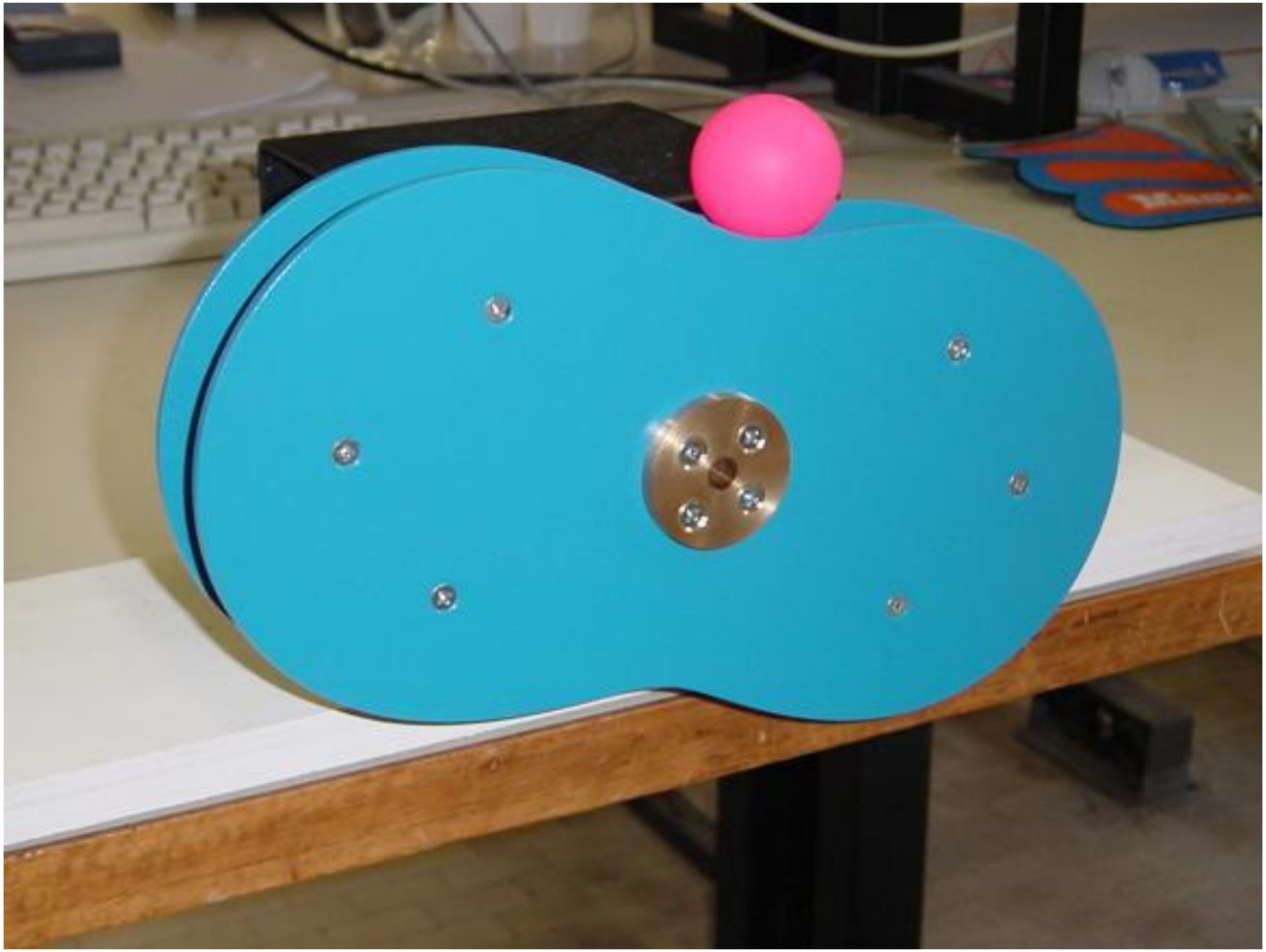
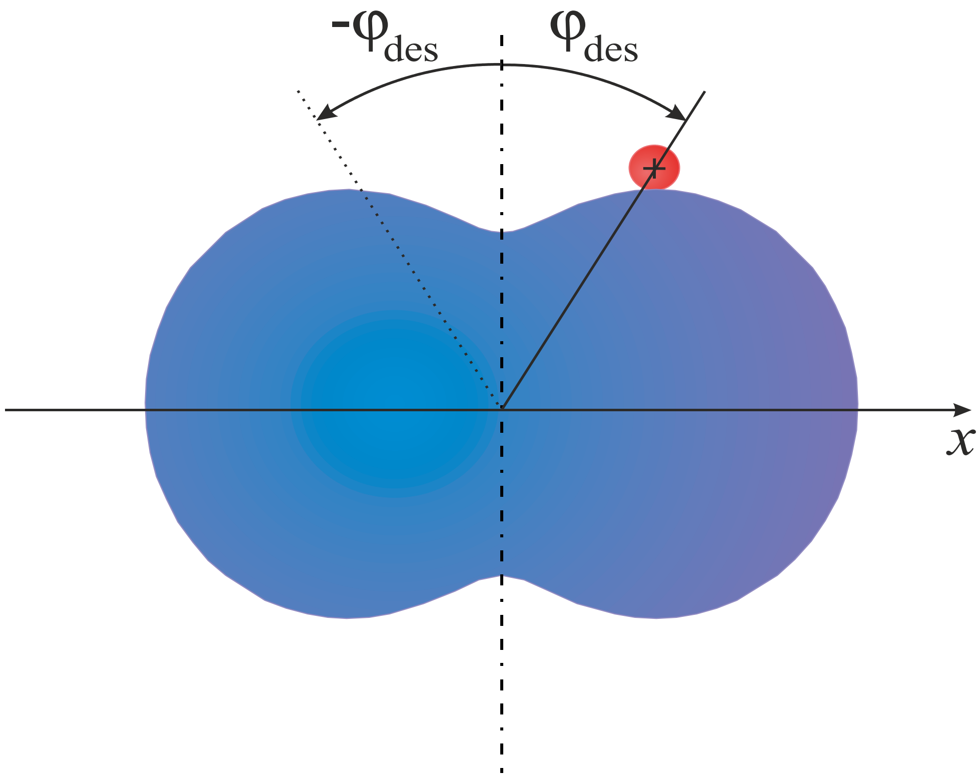
The "Butterfly" Robot



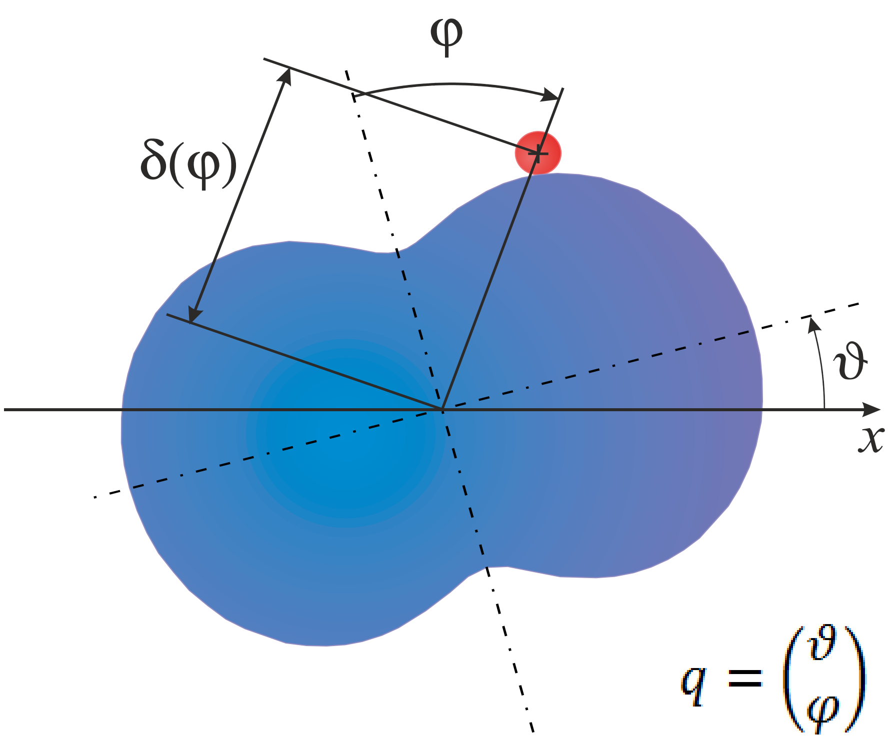
Problem formulation

- The ball moves from any point of the deep to the unstable point
- The ball oscillates (follow a desired periodic trajectory) around several critical points (stable and unstable equilibriums) with a particular desired frequency and magnitude.



Dynamics

- Ball is a point mass sliding without dissipation



$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + e(q) = \begin{pmatrix}\tau \\ 0\end{pmatrix}$$

$$B(q) = \begin{pmatrix} J + m\delta^2 & -m\delta^2 \\ -m\delta^2 & m\left(\delta^2 + m\delta'^2\right) \end{pmatrix}$$

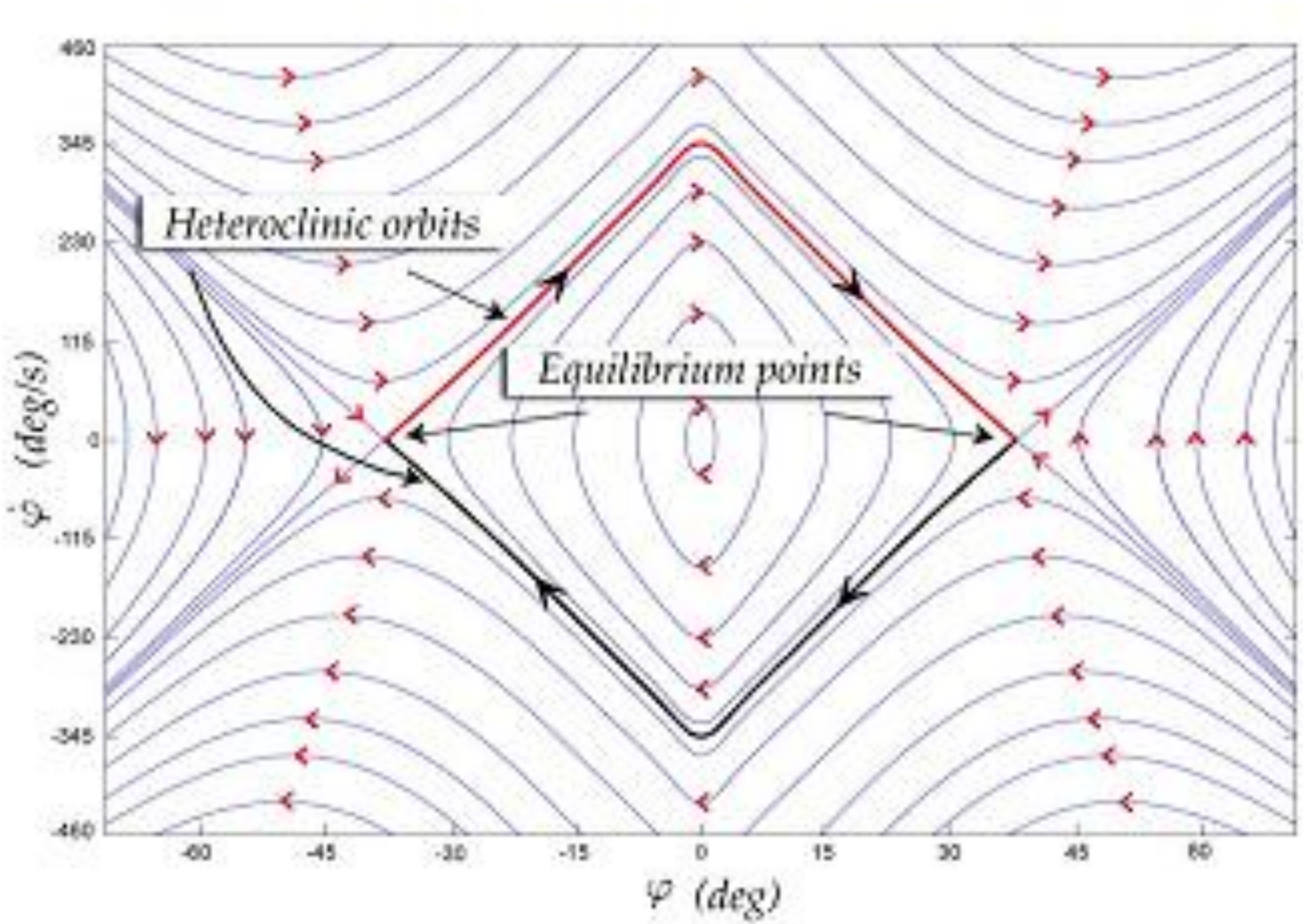
$$C(q,\dot{q}) = m\delta' \begin{pmatrix} \delta\dot{\varphi} & \delta(\dot{\vartheta} - 2\dot{\varphi}) \\ -\delta\,\dot{\vartheta} & \dot{\varphi}(\delta + \delta'') \end{pmatrix}$$

$$e(q) = \begin{pmatrix} -\delta\sin(\vartheta - \varphi) \\ \delta\sin(\vartheta - \varphi) + \delta'\cos(\vartheta - \varphi) \end{pmatrix}$$

$$q = \begin{pmatrix}\vartheta \\ \varphi\end{pmatrix} \qquad \delta(\varphi) = a(1 - b\cos^2\varphi)$$

- Underactuated mechanical system

Energy Based Method



Virtual Constraints Approach

- Geometric relation between generalized coordinates
  $\vartheta = \Theta(\varphi)$
- Periodic solution of obtained system
  $\varphi = \varphi(\varphi_0, \dot{\varphi}_0, t)$
- Controller for stabilize chosen constraint and trajectory

Choose Virtual Constraint

- It is easy to consider linear VC:

$$\vartheta = \vartheta_{des} + k(\varphi - \varphi_{des})$$

- Substituting to the source system, obtain:

$$\alpha(\varphi)\ddot{\varphi} + \beta(\varphi)\dot{\varphi}^2 + \gamma(\varphi) = 0$$

$$\alpha(\varphi) = \delta^2(1-k) + \delta'^2$$

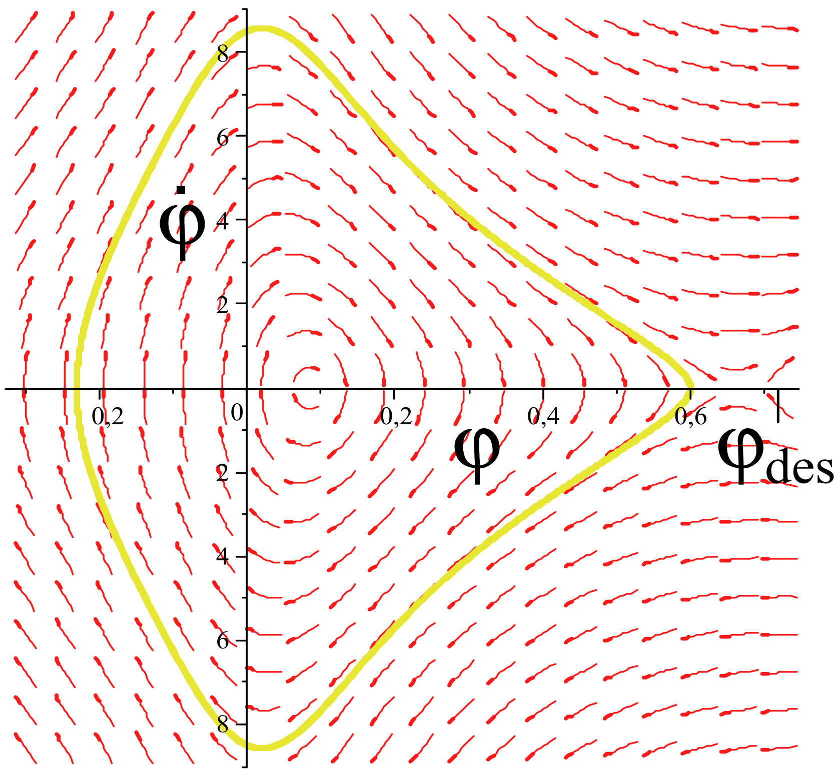
$$\beta(\varphi) = \delta'\delta'' + \delta\delta'(1-k^2)$$

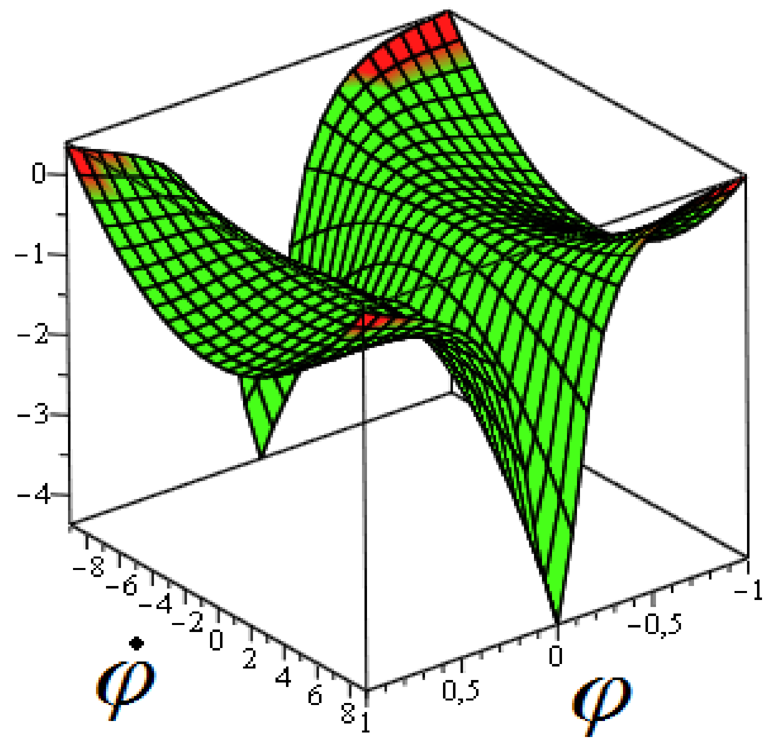
$$\gamma(\varphi) = g\delta'\cos\big(\varphi - \vartheta_{des} - k(\varphi - \varphi_{des})\big) -$$

$$-g\delta\sin\left(\varphi - \vartheta_{des} - k(\varphi - \varphi_{des})\right)$$

Particular Solution

- The distance between trajectory and equilibrium is a small (attractive zone of LQR regulator)
- The ball slides without any separations





Stabilize constraint

- virtual constraint error:

$$y = \vartheta - \Theta(\varphi) = \vartheta - \vartheta_{des} - k(\varphi - \varphi_{des})$$

- dynamics equations in newly coordinates:

$$q = \begin{pmatrix} y + \Theta(\varphi) \\ \varphi \end{pmatrix}$$

$$\begin{pmatrix} \ddot{y} \\ \ddot{\varphi} \end{pmatrix} = \underbrace{L^{-1}B^{-1}}_{K(y,\varphi)} \begin{pmatrix} \tau \\ 0 \end{pmatrix} - \underbrace{L^{-1}B^{-1}\left[CL\begin{pmatrix} \dot{y} \\ \dot{\varphi} \end{pmatrix} + G\right]}_{R(y,\varphi,\dot{y},\dot{\varphi})}$$

- introduce new virtual control law

$$\ddot{y} = K_{1,1}(y,\varphi) + R_1(y,\varphi,\dot{y},\dot{\varphi}) = \upsilon$$

Stabilize trajectory

- distance:

$$I(\varphi, \dot{\varphi}, \varphi_0, \dot{\varphi}_0)$$

$$= \dot{\varphi}^2 - \psi(\varphi_0, \varphi)\left({\dot{\varphi}_0}^2 - \int_{\varphi_0}^{\varphi} \psi(s, \varphi_0) \frac{2\gamma(s)}{\alpha(s)} ds \right)$$

$$\psi(\varphi_0, \varphi) = \exp\left\{ \int_{\varphi_0}^{\varphi} 2 \frac{\beta(s)}{\alpha(s)} ds \right\}$$

- derivative along DE $\alpha(\varphi)\ddot{\varphi} + \beta(\varphi)\dot{\varphi}^2 + \gamma(\varphi) = f$

$$\dot{I} = \frac{2\dot{\varphi}}{\alpha(\varphi)}(f - \beta(\varphi) I)$$

$$\alpha(\varphi)\ddot{\varphi} + \beta(\varphi)\dot{\varphi}^2 + \gamma(\varphi) = f =$$

$$\underbrace{\delta^2}_{g_{\upsilon}(\varphi,y)} \ \upsilon + \underbrace{\left(\delta\delta'\dot{y} + 2\delta\delta' k\dot{\varphi}\right)}_{g_{\dot{y}}(\varphi,y,\dot{\varphi},\dot{y})} \dot{y} -$$

$$\underbrace{-g\left(\delta'\sin\left(\varphi - \Theta(\varphi) - \frac{y}{2}\right) + \delta\cos\left(\varphi - \Theta(\varphi) - \frac{y}{2}\right)\right)\operatorname{sinc}\left(\frac{y}{2}\right)}_{g_{y}(\varphi,y,\dot{\varphi},\dot{y})} y$$

$$\alpha(\varphi)\ddot{\varphi} + \beta(\varphi)\dot{\varphi}^2 + \gamma(\varphi) = g_{\upsilon}\upsilon + g_{\dot{y}}\dot{y} + g_{y}y$$

Controller Design

$$\dot{\zeta} = \overline{A}(\varphi, y, \dot{\varphi}, \dot{y})\zeta + \overline{b}(\varphi, y, \dot{\varphi}, \dot{y})\upsilon \qquad \zeta = \begin{pmatrix} I \\ y \\ \dot{y} \end{pmatrix}$$

$$\overline{A}(\varphi, y, \dot{\varphi}, \dot{y}) = \begin{pmatrix} -\frac{2\dot{\varphi}}{\alpha}\beta & \frac{2\dot{\varphi}}{\alpha}g_y & \frac{2\dot{\varphi}}{\alpha}g_{\dot{y}} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\overline{b}(\varphi, y, \dot{\varphi}, \dot{y}) = \begin{pmatrix} \frac{2\dot{\varphi}}{\alpha}g_\upsilon \\ 0 \\ 1 \end{pmatrix}$$

Transversal Linearization Approach

- Consider the nonstationary linear system

$$\dot{\zeta} = A(t)\zeta + b(t)\upsilon$$

- where A(t) and b(t) evaluated long the desired trajectory
- It is well-known the optimal controller for such system:

$$\upsilon = -\Gamma^{-1} b(t)^T R(t)\zeta$$

- where R(t) is a solution if the Matrix Riccati Differential Equation (MRDE)

$$\dot{R}(t) + A(t)^T R(t) + R(t)A(t) + G$$
$$= R(t)b(t)\Gamma^{-1} b(t)^T R(t)$$

- Ad-hoc modification of the control law yields stabilizing controller for the source system

$$\upsilon = -\Gamma^{-1}\bar{b}(\varphi, y, \dot{\varphi}, \dot{y})^T R(t)\zeta$$

- To prove this, consider the Lyapunov function

$$V(\zeta, t) = \zeta^T R(t)\zeta$$

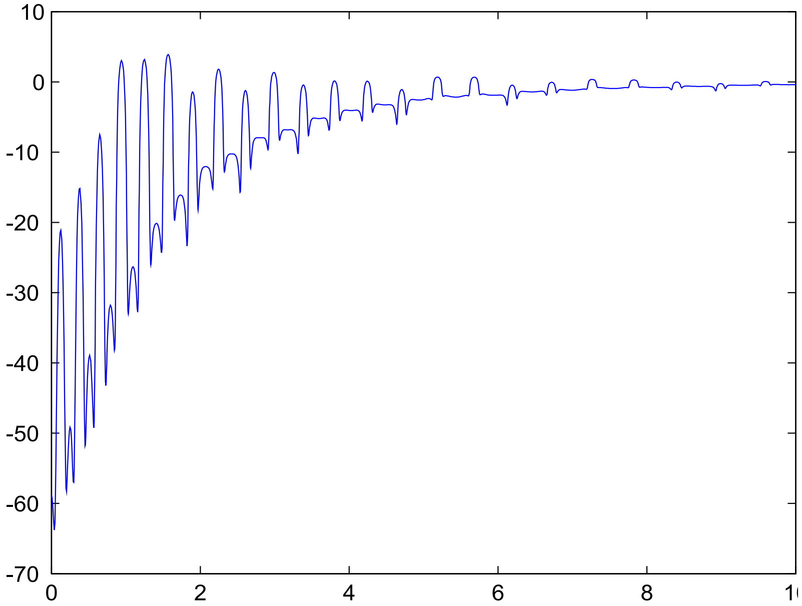
- derivative of it along the system equations:

$$\dot{V} = \zeta^T\left(-G - R(t)b(t)\Gamma^{-1}b(t)^T R(t) + \Delta(t, \varphi, y, \dot{\varphi}, \dot{y})\right)\zeta$$

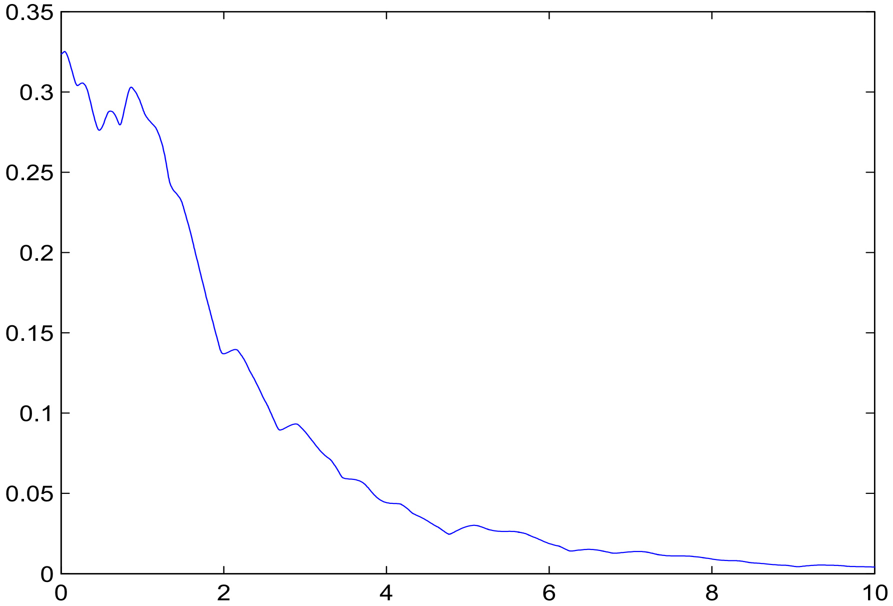
- it is not hard to show that in a neighborhood of the desired trajectory the following holds:

$$\Delta(\cdot) < \mathrm{G}$$

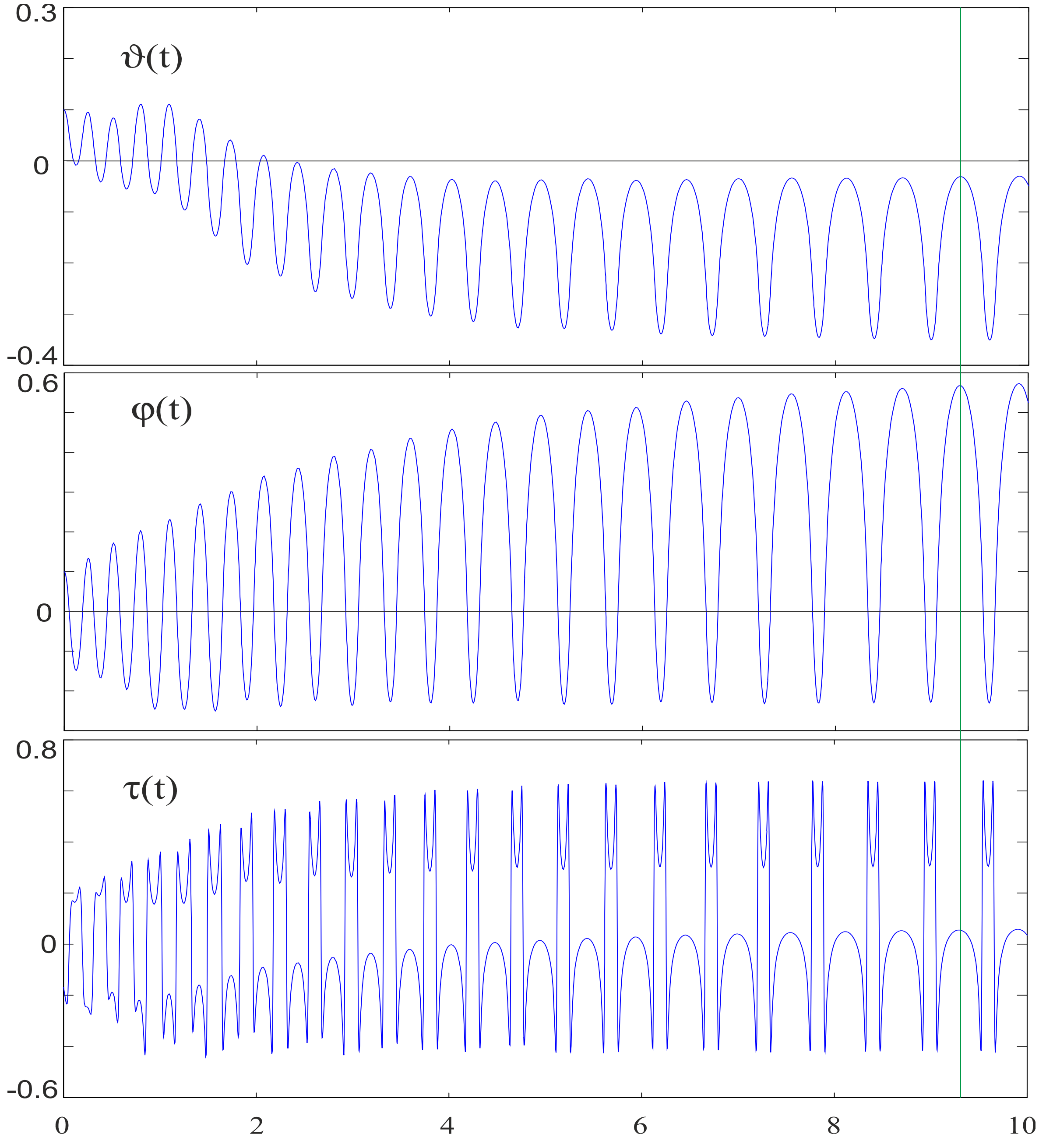
Simulation Results

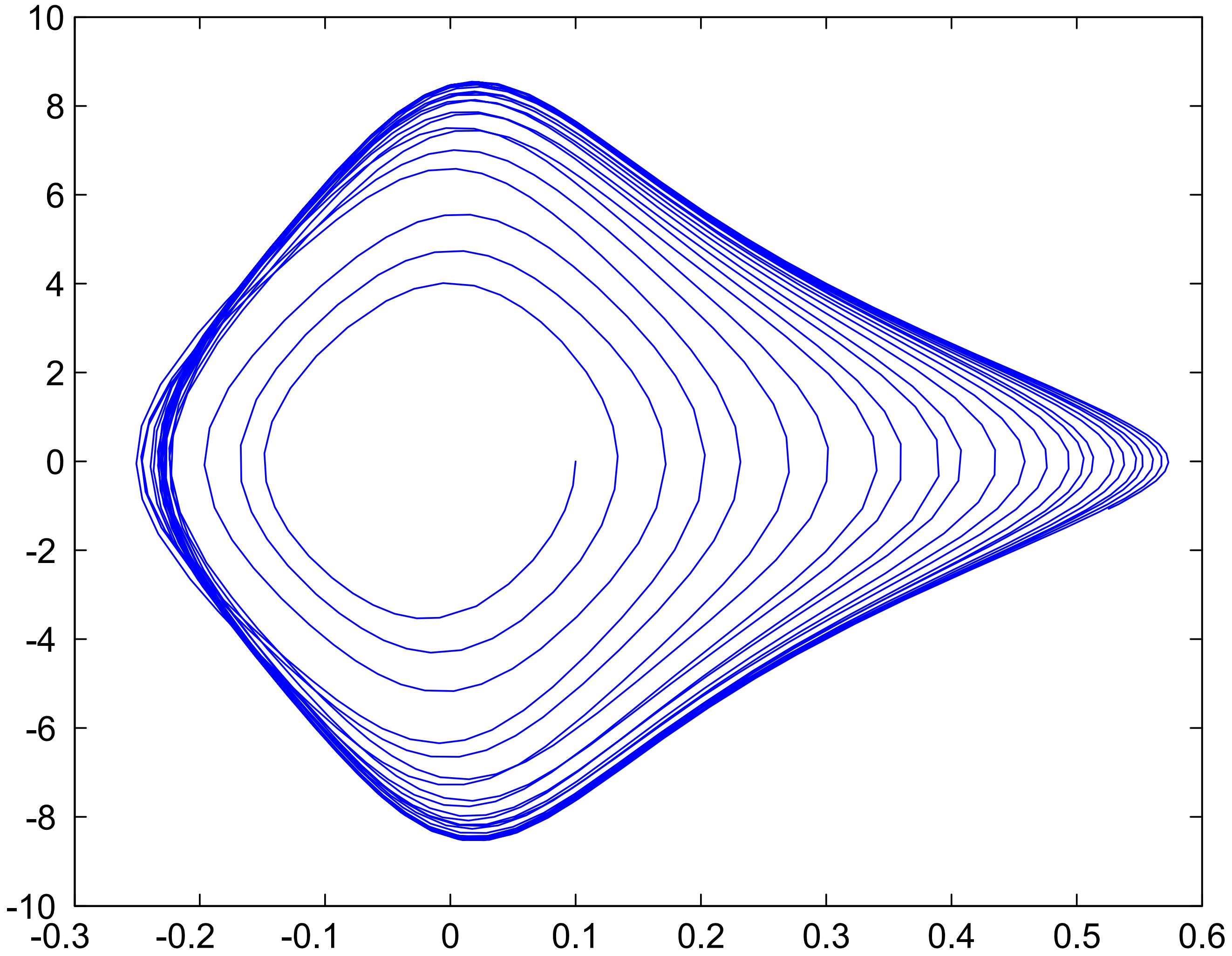


Virtual Constraint error



Trajectory error


0.3
ϑ(t)
0
-0.4
0.6
φ(t)
0
0.8
τ(t)
0
-0.6
0
2
4
6
8
10



10
8
6
4
2
0
-2
-4
-6
-8
-10
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6

Thank you for your attention!