Real algebraic geometry for the study of *D*-decomposition

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D-decomposition: problem statement.

Object

Let $P(s, p_1, \ldots, p_n)$ be a family of polynomials on one variable s depending on n real parameters.

Goal

The study of stability domains of $P(s)$ in the parameter space $\mathbb{K} = \{(p_1, \ldots, p_n)\}$.

Methodology

The study of partition of $\mathbb{K}$ by variety parametrized by $\Re(P(j\omega, p_1, \ldots, p_n)), \Im(P(j\omega, p_1, \ldots, p_n))$ and the leading term of $P(s)$

Motivation

$P(s, p_1, \ldots, p_n)$ is a characteristic polynomial of linear system.

D-decomposition: main examples.

D-decomposition for polynomials.

Affine family of polynomials could be understood as a characteristic polynomial of SISO system depending on parameters

$$P_0(s) + \sum_{i=1}^{n} p_i P_i(s).$$

D-decomposition for matrices.

We have to construct a stable linear output feedback for a system A, B, C. This leads to a construction of$D$-decomposition for a family $\chi_{A+BKC}(s, k_i j))$. Here χ is a characteristic polynomial, K- parameter matrix, $k_i j$- its elements.

D-decomposition for discrete objects

Stability analysis for discrete object could be reduced to the continous case using transformation below:
$P(s, p_1, \ldots p_n) \mapsto (s-1)^{\deg P} P(\frac{s+1}{s-1}, p_1, \ldots, p_n)$.

History of D-decomposition

- First example: I. Vyshnegradskii, 1876.
- Prehistory: Frazer R.A., Duncan W.J. 1929; A.A. Andronov, A.G. Mayer, A.A. Sokolov 1946.
- Formulation: Yu.I. Neimark, end of 1940-s; D. Mitrovic, 1959.
- Nonlinear and multiparametric cases: S.H. Lehnigk, D. Siljak. 2-nd half of 1960-s.
- Robust D-decomposition: Yu.I. Neimark, N.P. Petrov, B.T. Polyak. 1-st half of 1990-s; A.A. Tremba, the middle of 2000-s.
- Estimates for a number of regions of D-decomposition in 1-dimensional case and some classes of 2-dimensional families. E.N. Gryazina, B.T. Polyak the middle of 2000-s.
- Algebraic methods for D-decomposition: E.A. Jonckheere, 2-nd half 1990-s. A.D. Bruno et al. 2010-s.
- Estimates for the number of regions of D-decomposition in general 2- and n-dimensional cases. Current report.

Algebraic geometry and control theory: history

- "Global" linear systems theory. Similarity action: R.E. Kalman, R.Brockett, M. Hazewinkel, C.I. Byrnes, N.E. Hurt, C.Martin, A. Tannenbaum, P. Falb et. al. from 1976;
- Pole placement problem: C.I. Byrnes, R.Brockett, E.Sontag, M. Heymann, R. Hermann, H.Kimura, J. Rosenthal... :et.al. from 1978
- In Russia and former USSR: V.E. Belozerov, N.I. Osetinskii, V.Lomadze
- Computational real algebraic geometry in control(B.D.O. Anderson, P. Dorato, M.Jirstrand, A.Neubacher, H.Anai from1975 г.)

Real algebraic geometry and control theory

- Quantifier elimination for the linear output feedback stabilisation problem B.D.O. Anderson, N.K. Bose, E.I.Jury(1975)
- New birth of the field: M. Jirstrand, P.Dorato, R.Young, E.A. Jonckheere, A.Neubacher. (1996-1997).
- Quantifier elimination for stability testing of PDE H.Hong, R.Liska, P.Steinberg (1996)
- Robust control synthesis with quantifier elimination based on Sturm-Habicht sequence, Symbolic-Numeric CAD. Risa/Asir and Maple packages. H.Anai, S.Hara, H.Iwane, H.Yanami, H.Yokoyama second half of 2000-s.
- Real algebraic geometry for asymptotic stability analysis of nonlinear systems Z.She, B.Xue, Z.Zheng. 2011.
- Topology of real algebraic varieties for the study of D-decomposition. Current report.

Main result: n-dimensional case

Theorem 1.

Let $P(s, p_1, \dots, p_n)$ be a complex(in particular, may be real) polynomial of degree t on s and of degree d on all p_i together. Suppose that $\Re(P(j\omega, p_1, \dots, p_n))$ and $\Im(P(j\omega, p_1, \dots, p_n))$, do not have any common divisors non-trivially depending on ω.
Then the number of regions of D-decomposition is no greater than

$$6(4td + 4d)^n,$$

Main result: 2-dimensional case

Theorem 2

Let $P(s, r, p)$ be a polynomial of degree t with complex(in particular, possibly, real) coefficients.polynomially depending from two real parameters r, p or one complex parameter of a form $r + jp$. Denote by d its maximal degree as polynomial on variables r and p together. Suppose that $\Re(P(j\omega, r, p))$ and $\Im(P(j\omega, r, p))$, as complex polynomials does not have any common divisors that nontrivially depends on ω. Then the number of regions of D-decomposition is no greater than

$$\frac{q^2 + q + 2}{2},$$

with $q = 2td + 2d$.

Geometry of a proof

- Equations $\Re(P(j\omega, p_1, \ldots, p_n)) = 0$, $\Im(P(j\omega, p_1, \ldots, p_n)) = 0$ define a real algebraic variety Q in $\mathbb{K} \times \mathbb{R}$.
- Project Q onto $\mathbb{K}$.
- Projection of Q is semialgebraic, i.e. it is given by a finite set of polinomial equations and inequalities.
- $\Re(P(j\omega, p_1, \ldots, p_n))$, $\Im(P(j\omega, p_1, \ldots, p_n))$ have no common divisors nontrivially depending on ω, therefore $dimQ < dim\mathbb{K}$.
- Extend Q to a variety T.
- Estimate the number of regions of $\mathbb{K} \setminus T \cup L$, where L is defined by the leading term of $P(s, p_1, \ldots, p_n)$.

Algebra of a proof

- Polynomials $\Re(P(j\omega, p_1, \ldots, p_n)) = 0$, $\Im(P(j\omega, p_1, \ldots, p_n)) = 0$ define an ideal in the ring $\mathbb{R}[\omega, p_1, \ldots, p_n]$.
- We need to eliminate ω.
- Take an elimination ideal (generated by Groebner basis with some special ordering of monomials).
- Take its nonzero element(e.g. resultant). Its zeros break $\mathbb{K}$ in a parts. Their number is no greater than the number of parts generated by the basis of elimination ideal.
- Estimate the number of components of $\mathbb{K}$ after breakup by a polynomial of a fixed degree using lemma on "cutting cabbage" in 2-dimensional case and Warren inequality in n-dimensional one.

Math inside

Proofs are rather simple but they are connected with a number of topics:

- Elimination theory.
- Plane real algebraic curves: Harnack theorem, 16-th Hilbert problem, Harnack problem.
- Real algebraic geometry: quantifier elimination, Tarski-Seidenberg theorem, cylindrical decomposition, Komessati-Petrovsky-Oleinik inequality, Warren inequality.

Mathematical foundations: Harnack Theorem(1876)

Real plane algebraic curve irreducible algebraic curve of degree n has no more than

$$\frac{(n-1)(n-2)}{2}+1$$

connected components and this bound is sharp.

Harnack and Bezout theorems

Mathematical foundations: Harnack Theorem(1876)

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Mathematical foundations: Bezout Theorem

Let f, g be algebraic curves of degrees m, n respectively. Then they have either common irreducible components or number of their intersection points is no greater than mn.

Lemma on “cutting cabbage”

Let X be the plane real (possibly singular and reducible) affine algebraic curve of degree n. Then its complement in $\mathbb{R}^2$ consists of no more than $\frac{n^2+n+2}{2}$ connected components. This bound is sharp and it is reached on union of lines in general position.

Scheme of the proof

- Local structure of intersection points of irreducible component of curve
- "Globalisation" of local structure: every intersection point adds no more than one new region.
- Application of Harnack and Bezout estimates.
- Solution of an extremal problem.
- Realizability of the estimate: the case of lines in general position.

3-dimensional generalisations

There is no sharp Harnack-like theorems for higher-dimensional real algebraic varieties.
Moreover, we can easily show that the case of hyperplanes in general position even in 3-dimensional case do not give a maximal number of regions.

Bihan's asymptotics(1998)

Maximal number of connected components of real projective algebraic surface of degree q is equivalent as $q \to \infty$ to dq^3, where $d \in [\frac{13}{36}, \frac{5}{12}]$.

Inequality(consequence of Comessatti-Petrovsky-Oleinik and Smith-Thom inequalities

Maximal number of components of real algebraic surface is no greater than $\frac{5}{12}q^3 - \frac{3}{2}q^2 + \frac{25}{12}q$.

But q planes could divide the space only in $\frac{q^3+5q+6}{6}$ parts.

n-dimensional bound. Warren inequality.

Warren inequality(1968)

Let $r_1, \ldots, r_m$ be real polynomials in n variables, each of degree d or less. Let $N(r_i)$ be the set of zeros of r_i. The number of connected components of the set $\mathbb{R}^n \setminus \cup_{i=1}^m N(r_i)$ does not exceed $n = \sum_{k=0}^n 2(2d)^n 2^k C_{m,k}$, where $C_{m,k}$ the usual binomial coefficient, except that is $C_{m,k} = 0$ for $m < k$.

n-dimensional bound could, possibly, became sharper. The results of S. Basu, R.Pollack, M-F. Roy(2009) indicates that possibly the first bound could be sharpened by forgetting 2^k.

Computational procedures

In order to study 2-dimensional D-decompositions with Maple 14 we have used the following scheme.

1. Convertion of family of matrices to the family of polynomials $P(s, r, p)$, if needed.
2. If we think about our family as a discrete-time object then we convert it to continuous-time using transformation $P(s) \mapsto (s-1)^{\deg P} P(\frac{s+1}{s-1})$.
3. Computing an equation defining most of irreducible components of an algebraic curve, containing border of D-decomposition using command *EliminationIdeal* for an ideal generated by $\Re(P(j\omega, r, p))$ and $\Im(P(j\omega, r, p))$ or eliminating ω using resultants.
4. if $P(s, r, p)$ has a real leading term then we add an irreducible component generated by its leading coefficient $a_{\deg P}(r, p)$ else we add $\Re^2(a_{\deg P}(r, p)) + \Im^2(a_{\deg P}(r, p))$. We get an algebraic curve containing the border of D-decomposition.
5. Using *PartialCylindricalAlgebraicDecomposition*, we obtain a point cloud that contains at least one point from every region of D-decomposition.

Family $s^6 + (r + jp)s^5 + \frac{3}{2}$., as a discrete-time object: Explicit equation of D-decomposition border of a degree 10 :

$$9216p^{10} + 46080p^8 r^2 + 92160p^6 r^4 + 92160p^4 r^6 + 46080p^2 r^8 + 9216r^{10} - \\ -94464p^8 - 377856p^6 r^2 - 566784p^4 r^4 - 377856p^2 r^6 - 94464r^8 + 301440p^6 + \\ +683136p^4 r^2 + 1051776p^2 r^4 + 276864r^6 - 309600p^4 - 619200p^2 r^2 - 309600r^4 + \\ +122500p^2 + 122500r^2 - 15625 = 0$$

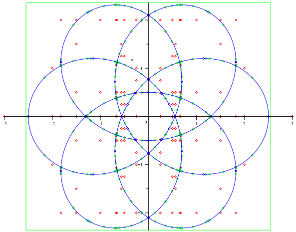
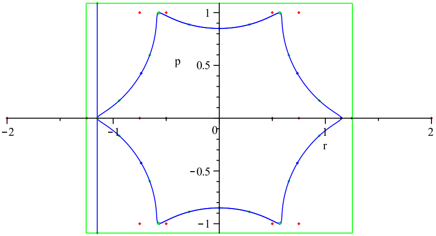


Рис.: D-decomposition and point cloud from regions of D-decomposition for the family $s^6 + (r + jp)s^5 + \frac{3}{2}$.

Example 2.

Family $z^6 + (r + jp)z^5 + \frac{3}{20}$, as a discrete-time object: Equation that define a border of D-decomposition:

$$
\begin{aligned}
&9,216 \cdot 10^{13} p^{10} + 4,608 \cdot 10^{14} p^8 r^2 + 9,216 \cdot 10^{14} p^6 r^4 + 9,216 \cdot 10^{15} p^4 r^6 + \\
&+ 4,608 \cdot 10^{14} p^2 r^8 + 9,216 \cdot 10^{13} r^{10} + 8,1792 \cdot 10^{13} p^8 + 3,27168 \cdot 10^{14} p^6 r^2 + \\
&+ 4,90752 \cdot 10^{14} p^4 r^4 + 3,27168 \cdot 10^{14} p^2 r^6 + 8,1792 \cdot 10^{13} r^8 + \\
&+ 1,30276416 \cdot 10^{15} p^6 - 1,821010752 \cdot 10^{16} p^4 r^2 + 1,865389248 \cdot 10^{15} p^2 r^4 - \\
&- 1,15483584 \cdot 10^{15} r^6 + 6,82460784 \cdot 10^{11} p^4 + 1,364921568 \cdot 10^{14} p^2 r^2 + \\
&+ 6,82460784 \cdot 10^{13} r^4 + 4,160322828658 \cdot 10^{15} p^2 + 4,160322828658 \cdot 10^{15} r^2 - \\
&- 3,573226485213841 \cdot 10^{15} = 0.
\end{aligned}
$$



Example 3.

Ouput feedback problem of type $K = \begin{pmatrix} -r & p \\ p & r \end{pmatrix}$ for a continous-time system given by matrices below:

$$A = \begin{pmatrix} 79 & 20 & -30 & -20 \\ -41 & -12 & 17 & 13 \\ 167 & 40 & -60 & -38 \\ 33,5 & 9 & -14,5 & -11 \end{pmatrix}, B = \begin{pmatrix} 0,219 & 0,9346 \\ 0,047 & 0,3835 \\ 0,6789 & 0,5194 \\ 0,6793 & 0,831 \end{pmatrix},$$

$$C = \begin{pmatrix} 0,0346 & 0,5297 & 0,0077 & 0,0668 \\ 0,0535 & 0,6711 & 0,3834 & 0,4175 \end{pmatrix}.$$

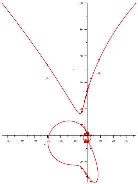


Рис.: Minimal algebraic curve containing the border of D-decomposition and point cloud

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