Logarithmic criterion on formation of the security portfolio

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1. Notation and main assumptions

- C_0 is the investor's initial capital
- b_0 is return of the risk-free asset
- X_1 and X_2 are returns of first and second risky assets
  $X_1 \sim \mathcal{R}[-1, 1+2m_1], X_2 \sim \mathcal{R}[-1, 1+2m_2]$, where
  $m_2 > m_1 > b_0 > 0$ X_1 and X_2 are independent random variables
- u_0 is a part of C_0 invested in the risk-free asset
- u_1 is a part of C_0 invested in the first risky asset
- u_2 is a part of C_0 invested in the second risky asset

Total amount of investments may be less, than the investor's initial capital, short-sales are not allowed.
The investor prefers an asset with higher mean return and higher risk i.e. variance respectively.

2. The statement of the problem

The set of admissible strategies is

$$U \overset{def}{=} \{u : u_0 + u_1 + u_2 \leq 1, u_2 \geq u_1 \geq 0, u_0 \geq 0\}. \qquad (1)$$

As the utility function we consider capital growth rate

$$\Phi(u, X) \overset{def}{=} \ln(C_0(1 + u_0 b_0 + u_1 X_1 + u_2 X_2)).$$

As the optimality criterion we use average capital growth rate

$$\Phi_0(u) \overset{def}{=} \mathbf{M}\left[\ln(C_0(1 + u_0 b_0 + u_1 X_1 + u_2 X_2))\right] \to \max_{u \in U}.$$

3. Standard form of the objective

Whereas

$$\Phi_0(u) = \mathbf{M}\left[\ln(C_0(1 + u_0 b_0 + \hat{X}_1 - u_1 + \hat{X}_2 - u_2))\right],$$

where $\hat{X}_1 \sim \mathcal{R}[0, 2u_1(1+m_1)]$, $\hat{X}_2 \sim R[0, 2u_2(1+m_2)]$, then by the convolution densities

$$f_Y(y) \stackrel{def}{=} \int\limits_{-\infty}^{+\infty} f_{\hat{X}_1}(x) f_{\hat{X}_2}(y-x)dx$$

we have

$$\Phi_0(u) = \mathbf{M}\left[\ln(C_0(1 + u_0 b_0 - u_1 - u_2 + Y))\right] = \\ = \int\limits_{-\infty}^{+\infty} f_Y(y) \ln(C_0(1 + u_0 b_0 - u_1 - u_2 + y))dy. \quad (2)$$

4. Concavity of the objective

Lemma 1

The random variable Y has density

$$f_Y(y) = \begin{cases} \frac{y}{ab}, & 0 \le y \le a, \\ \frac{a}{ab}, & a \le y \le b, \\ \frac{a+b-y}{ab}, & b \le y \le a+b, \\ 0, & y \le 0, \\ 0, & y \ge a+b, \end{cases}$$

where $a \stackrel{def}{=} 2u_1(1+m_1), b \stackrel{def}{=} 2u_2(1+m_2)$.

Lemma 2

The objective $\Phi_0(u)$, *defined according to (2), is concave on the convex set* U, *defined according to (1).*

5. The objective maximization problem

Thus, our goal is to solve

$$u^* = \arg\max_{u \in U} \Phi_0(u), \Phi_0^* = \max_{u \in U} \Phi_0(u), \tag{3}$$

where function $\Phi_0(u)$ is defined by (2). The set U for problem (3) we can replace by

$$\hat{U} \stackrel{def}{=} \{u : u_0 + u_1 + u_2 = 1, u_2 \geq u_1 \geq 0, u_0 \geq 0\}.$$

Then problem (3) is equivalent to the problem

$$u^* = \arg\max_{u \in \hat{U}} \Phi_0(u), \Phi_0^* = \max_{u \in \hat{U}} \Phi_0(u).$$

Theorem 1

The objective, defined according to (2), for all $u \in \hat{U}$ such that $u_i > 0$, is

$$\Phi_0(u) = \frac{1}{4u_1u_2\hat{m}_1\hat{m}_2}\left[-\frac{(2u_1\hat{m}_1 + u_0\hat{b}_0)^2\ln(2u_1\hat{m}_1 + u_0\hat{b}_0)}{2} + \right.$$
$$+\frac{(2u_1\hat{m}_1 + 2u_2\hat{m}_2 + u_0\hat{b}_0)^2\ln(2u_1\hat{m}_1 + 2u_2\hat{m}_2 + u_0\hat{b}_0)}{2} -$$
$$\left. -\frac{(2u_2\hat{m}_2 + u_0\hat{b}_0)^2\ln(2u_2\hat{m}_2 + u_0\hat{b}_0)}{2} + \frac{(u_0\hat{b}_0)^2}{2}\ln(u_0\hat{b}_0)\right] + \ln(C_0) - \frac{3}{2}, \tag{4}$$

where $\hat{m}_1 \stackrel{def}{=} 1 + m_1$, $\hat{m}_2 \stackrel{def}{=} 1 + m_2$, $\hat{b}_0 \stackrel{def}{=} 1 + b_0$.

7. The explicit form of the objective

Lemma 3

At points $(0, u_1, u_2) \in \hat{U}$ *objective (2) is continuous and equal to*

$$\Phi_0(0, u_1, u_2) = \ln(C_0) - \frac{3}{2} + \frac{1}{4u_1 u_2 \hat{m}_1 \hat{m}_2}\left[-\frac{(2u_2\hat{m}_2)^2 \ln(2u_2\hat{m}_2)}{2} + \right.$$

$$\left. + \frac{(2u_1\hat{m}_1 + 2u_2\hat{m}_2)^2 \ln(2u_1\hat{m}_1 + 2u_2\hat{m}_2)}{2} - \frac{(2u_1\hat{m}_1)^2 \ln(2u_1\hat{m}_1)}{2}\right]; \quad (5)$$

Lemma 3

at points $(u_0, 0, u_2) \in \hat{U}$ *objective (2) is continuous and equal to*

$$\Phi_0(u_0, 0, u_2) = \ln(C_0) - \frac{3}{2} + \frac{1}{4u_2\hat{m}_2}\Big[-2u_0\hat{b}_0 \ln(u_0\hat{b}_0) + +2\hat{m}_2 u_2 + 2(2u_2\hat{m}_2 + u_0\hat{b}_0)\ln(2u_2\hat{m}_2 + u_0\hat{b}_0)\Big]; \qquad (6)$$

at points $(1, 0, 0) \in \hat{U}$ *objective (2) is continuous and equal to*

$$\Phi_0(1, 0, 0) = \ln(C_0) + \ln(\hat{b}_0). \qquad (7)$$

We introduce sets:

$$U_\theta \overset{def}{=} \{u : u_0 + u_1 + u_2 = 1, u_2 - u_1 \geq 0, u_1 - \theta \geq 0, u_0 - \theta \geq 0\}, \quad (8)$$

$$\hat{U}_{\theta,0} \overset{def}{=} \{u : u_1 + u_2 = 1, u_2 - u_1 \geq 0, u_1 - \theta \geq 0\}, \quad (9)$$

$$\hat{U}_{\theta,1} \overset{def}{=} \{u : u_0 + u_2 = 1, u_2 - \theta \geq 0, u_0 - \theta \geq 0\}. \quad (10)$$

If maximum of function (4) attained on the boundary U_θ for sufficiently small θ, we seek maximum from maximum of function (5) on set (9), maximum of function (6) on set (10) and value (7). If obtained maximum larger than maximum of function (4), then this strategy will be optimal.

For $\theta = 10^{-5}$ and $C_0 = 10000$ we have

№	b_0	m_1	m_2	u_0^*	u_1^*	u_2^*
1	0.05	0.06	0.07	0.916969	0.028007	0.055023
2	0.05	0.15	0.2	0.447739	0.228295	0.323965
3	0.05	0.15	0.4	0.230428	0.196436	0.573135

Table 1. The security portfolio.