Detecting communities by voting model

Yurii Nesterov, CORE/INMA (UCL)

Traditional Mathematical School
June 22-29, 2014 (Moscow)

Outline

1. Compact data representations
2. Voting model
3. Behavior of political parties
4. Consecutive elections
5. Convergence

Compact data representations

Compact data representations

Different models:

Compact data representations

Different models:

- Model reduction (Control theory)

Compact data representations

Different models:

- Model reduction (Control theory)
- Clustering (Image processing, medical applications)

Compact data representations

Different models:

- Model reduction (Control theory)
- Clustering (Image processing, medical applications)
- Community detection (Networks)

Compact data representations

Different models:

- Model reduction (Control theory)
- Clustering (Image processing, medical applications)
- Community detection (Networks)

Common drawbacks:

Compact data representations

Different models:

- Model reduction (Control theory)
- Clustering (Image processing, medical applications)
- Community detection (Networks)

Common drawbacks:

- Non-uniqueness.

Compact data representations

Different models:

- Model reduction (Control theory)
- Clustering (Image processing, medical applications)
- Community detection (Networks)

Common drawbacks:

- Non-uniqueness.
- Many local minimums in the auxiliary optimization problems.

Compact data representations

Different models:

- Model reduction (Control theory)
- Clustering (Image processing, medical applications)
- Community detection (Networks)

Common drawbacks:

- Non-uniqueness.
- Many local minimums in the auxiliary optimization problems.
- Impossible to compare efficiency of different approaches.

Compact data representations

Different models:

- Model reduction (Control theory)
- Clustering (Image processing, medical applications)
- Community detection (Networks)

Common drawbacks:

- Non-uniqueness.
- Many local minimums in the auxiliary optimization problems.
- Impossible to compare efficiency of different approaches.

Main question:

Compact data representations

Different models:

- Model reduction (Control theory)
- Clustering (Image processing, medical applications)
- Community detection (Networks)

Common drawbacks:

- Non-uniqueness.
- Many local minimums in the auxiliary optimization problems.
- Impossible to compare efficiency of different approaches.

Main question:

Can we have data representation models with unique solution?

Voting model

Voting model

- N independent voters.

Voting model

- N independent voters.
- ℓ political parties.

Voting model

- N independent voters.
- ℓ political parties.

Assumption:

Voting model

- N independent voters.
- ℓ political parties.

Assumption: *The results of personal votes are random.*

Voting model

- N independent voters.
- ℓ political parties.

Assumption: *The results of personal votes are random.*

Voter i decides to vote for party k with probability $p_i^{(k)}$.

Voting model

- N independent voters.
- ℓ political parties.

Assumption: *The results of personal votes are random.*

Voter i decides to vote for party k with probability $p_i^{(k)}$.

Notation: $p_i = (p_i^{(1)}, \dots, p_i^{(\ell)})^T \in \Delta_\ell$.

Voting model

- N independent voters.
- ℓ political parties.

Assumption: *The results of personal votes are random.*

Voter i decides to vote for party k with probability $p_i^{(k)}$.

Notation: $p_i = (p_i^{(1)}, \ldots, p_i^{(\ell)})^T \in \Delta_\ell$. (Unknown)

Voting model

- N independent voters.
- ℓ political parties.

Assumption: *The results of personal votes are random.*

Voter i decides to vote for party k with probability $p_i^{(k)}$.

Notation: $p_i = (p_i^{(1)}, \dots, p_i^{(\ell)})^T \in \Delta_\ell$. (Unknown)

Matrix $P = (p_1, \dots, p_N) \in \mathbb{R}^{\ell \times N}$, is called the *voting matrix*.

Voting model

- N independent voters.
- ℓ political parties.

Assumption: *The results of personal votes are random.*

Voter i decides to vote for party k with probability $p_i^{(k)}$.

Notation: $p_i = (p_i^{(1)}, \dots, p_i^{(\ell)})^T \in \Delta_\ell$. (Unknown)

Matrix $P = (p_1, \dots, p_N) \in \mathbb{R}^{\ell \times N}$, is called the *voting matrix*.

Personal opinions: $v_i \in \mathcal{V} \subseteq \mathbb{R}^m$, $i = 1, \dots, N$, where $\mathcal{V}$ is a closed convex set (e.g. $\mathcal{V} = \mathbb{R}^m_+$).

Voting model

- N independent voters.
- ℓ political parties.

Assumption: *The results of personal votes are random.*

Voter i decides to vote for party k with probability $p_i^{(k)}$.

Notation: $p_i = (p_i^{(1)}, \dots, p_i^{(\ell)})^T \in \Delta_\ell$. (Unknown)

Matrix $P = (p_1, \dots, p_N) \in \mathbb{R}^{\ell \times N}$, is called the *voting matrix*.

Personal opinions: $v_i \in \mathcal{V} \subseteq \mathbb{R}^m$, $i = 1, \dots, N$, where $\mathcal{V}$ is a closed convex set (e.g. $\mathcal{V} = \mathbb{R}^m_+$). (Fixed)

Voting model

- N independent voters.
- ℓ political parties.

Assumption: *The results of personal votes are random.*

Voter i decides to vote for party k with probability $p_i^{(k)}$.

Notation: $p_i = (p_i^{(1)}, \dots, p_i^{(\ell)})^T \in \Delta_\ell$. (Unknown)

Matrix $P = (p_1, \dots, p_N) \in \mathbb{R}^{\ell \times N}$, is called the *voting matrix*.

Personal opinions: $v_i \in \mathcal{V} \subseteq \mathbb{R}^m$, $i = 1, \dots, N$, where $\mathcal{V}$ is a closed convex set (e.g. $\mathcal{V} = \mathbb{R}^m_+$). (Fixed)

Positions of parties: $x_k \in \mathcal{V}$, $k = 1, \dots, \ell$,

Voting model

- N independent voters.
- ℓ political parties.

Assumption: <u>*The results of personal votes are random.*</u>

Voter i decides to vote for party k with probability $p_i^{(k)}$.

Notation: $p_i = (p_i^{(1)}, \ldots, p_i^{(\ell)})^T \in \Delta_\ell$. (Unknown)

Matrix $P = (p_1, \ldots, p_N) \in \mathbb{R}^{\ell \times N}$, is called the *voting matrix*.

Personal opinions: $v_i \in \mathcal{V} \subseteq \mathbb{R}^m$, $i = 1, \ldots, N$, where $\mathcal{V}$ is a closed convex set (e.g. $\mathcal{V} = \mathbb{R}^m_+$). (Fixed)

Positions of parties: $x_k \in \mathcal{V}$, $k = 1, \ldots, \ell$, are <u>*flexible*</u>.

Voting model

- N independent voters.
- ℓ political parties.

Assumption: *The results of personal votes are random.*

Voter i decides to vote for party k with probability $p_i^{(k)}$.

Notation: $p_i = (p_i^{(1)}, \dots, p_i^{(\ell)})^T \in \Delta_\ell$. (Unknown)

Matrix $P = (p_1, \dots, p_N) \in \mathbb{R}^{\ell \times N}$, is called the *voting matrix*.

Personal opinions: $v_i \in \mathcal{V} \subseteq \mathbb{R}^m$, $i = 1, \dots, N$, where $\mathcal{V}$ is a closed convex set (e.g. $\mathcal{V} = \mathbb{R}^m_+$). (Fixed)

Positions of parties: $x_k \in \mathcal{V}$, $k = 1, \dots, \ell$, are *flexible*.

Notation: $X = (x_1, \dots, x_\ell) \in \mathbb{R}^{m \times \ell}$.

Voting model

- N independent voters.
- ℓ political parties.

Assumption: *The results of personal votes are random.*

Voter i decides to vote for party k with probability $p_i^{(k)}$.

Notation: $p_i = (p_i^{(1)}, \dots, p_i^{(\ell)})^T \in \Delta_\ell$. (Unknown)

Matrix $P = (p_1, \dots, p_N) \in \mathbb{R}^{\ell \times N}$, is called the *voting matrix*.

Personal opinions: $v_i \in \mathcal{V} \subseteq \mathbb{R}^m$, $i = 1, \dots, N$, where $\mathcal{V}$ is a closed convex set (e.g. $\mathcal{V} = \mathbb{R}^m_+$). (Fixed)

Positions of parties: $x_k \in \mathcal{V}$, $k = 1, \dots, \ell$, are *flexible*.

Notation: $X = (x_1, \dots, x_\ell) \in \mathbb{R}^{m \times \ell}$. (Unknown)

Discrete choice model

Discrete choice model

Logit model:

Discrete choice model

Logit model: discrete random choice for

$$\min_{1 \leq i \leq \ell} \{c^i + \epsilon_i\},$$

where ϵ_i are i.i.d. (double exponential) with deviation $\mu > 0$.

Discrete choice model

Logit model: discrete random choice for

$$\min_{1\le i\le \ell} \{c^i + \epsilon_i\},$$

where ϵ_i are i.i.d. (double exponential) with deviation $\mu > 0$.

Choice probabilities: $\pi^i(c) = e^{-c^i/\mu} / \left[\sum\limits_{j=1}^{\ell} e^{-c^j/\mu} \right]$.

Discrete choice model

Logit model: discrete random choice for

$$\min_{1 \le i \le \ell} \{ c^i + \epsilon_i \},$$

where ϵ_i are i.i.d. (double exponential) with deviation $\mu > 0$.

Choice probabilities: $\pi^i(c) = e^{-c^i/\mu} / \left[\sum\limits_{j=1}^{\ell} e^{-c^j/\mu} \right]$.

For $\mu \to 0$ we get deterministic rationality.

Optimization interpretation

Optimization interpretation

Define $\eta(x) = \sum\limits_{i=1}^{\ell} x^i \ln x^i$ (entropy function)

Optimization interpretation

Define $\eta(x) = \sum\limits_{i=1}^{\ell} x^i \ln x^i$ (entropy function)

It is strongly convex on Δ_ℓ in ℓ_1 norm with parameter one:

Optimization interpretation

Define $\eta(x) = \sum\limits_{i=1}^{\ell} x^i \ln x^i$ (entropy function)

It is strongly convex on Δ_ℓ in ℓ_1 norm with parameter one:

$$\langle \nabla^2 \eta(x) h, h \rangle = \sum_{i=1}^{\ell} \frac{(h^i)^2}{x^i} \geq \|h\|_1^2 \text{ for } x \in \Delta_\ell, \; h \in \mathbb{R}^{|ell}.$$

Optimization interpretation

Define $\eta(x) = \sum\limits_{i=1}^{\ell} x^i \ln x^i$ (entropy function)

It is strongly convex on Δ_ℓ in ℓ_1 norm with parameter one:

$$\langle \nabla^2 \eta(x) h, h \rangle = \sum_{i=1}^{\ell} \frac{(h^i)^2}{x^i} \geq \|h\|_1^2 \text{ for } x \in \Delta_\ell, \; h \in \mathbb{R}^{|ell}.$$

$$\text{Define } \theta_\mu(c) = \min_{\pi \in \Delta_\ell} \{\langle c, \pi \rangle + \mu \eta(\pi)\} = -\mu \ln \left(\sum_{i=1}^{\ell} e^{-c^i/\mu} \right).$$

Optimization interpretation

Define $\eta(x) = \sum\limits_{i=1}^{\ell} x^i \ln x^i$ (entropy function)

It is strongly convex on Δ_ℓ in ℓ_1 norm with parameter one:

$$\langle \nabla^2 \eta(x) h, h \rangle = \sum_{i=1}^{\ell} \frac{(h^i)^2}{x^i} \geq \|h\|_1^2 \text{ for } x \in \Delta_\ell, \; h \in \mathbb{R}^{|ell}.$$

Define $\theta_\mu(c) = \min\limits_{\pi \in \Delta_\ell} \{ \langle c, \pi \rangle + \mu \eta(\pi) \} = -\mu \ln \left(\sum\limits_{i=1}^{\ell} e^{-c^i/\mu} \right)$. Then

$$\pi(c) = \nabla \theta_\mu(c).$$

Optimization interpretation

Define $\eta(x) = \sum\limits_{i=1}^{\ell} x^i \ln x^i$ (entropy function)

It is strongly convex on Δ_ℓ in ℓ_1 norm with parameter one:

$$\langle \nabla^2 \eta(x) h, h \rangle = \sum_{i=1}^{\ell} \frac{(h^i)^2}{x^i} \geq \|h\|_1^2 \text{ for } x \in \Delta_\ell, \; h \in \mathbb{R}^{|ell}.$$

Define $\theta_\mu(c) = \min\limits_{\pi \in \Delta_\ell} \{ \langle c, \pi \rangle + \mu \eta(\pi) \} = -\mu \ln \left(\sum\limits_{i=1}^{\ell} e^{-c^i/\mu} \right)$. Then

$$\pi(c) = \nabla \theta_\mu(c).$$

Important: $\|\nabla \theta_\mu(c_1) - \nabla \theta_\mu(c_2)\|_1 \leq \frac{1}{\mu} \|c_1 - c_2\|_\infty \quad \forall c_1, c_2 \in \mathbb{R}^\ell.$
(see later)

Probabilistic model

Probabilistic model

Let us fix some norm $\| \cdot \|$ in $\mathbb{R}^m$.

Probabilistic model

Let us fix some norm $\| \cdot \|$ in $\mathbb{R}^m$.

Intuition: the bigger the is the distance between the opinion of a voter and position of a party,

Probabilistic model

Let us fix some norm $\| \cdot \|$ in $\mathbb{R}^m$.

Intuition: the bigger the is the distance between the opinion of a voter and position of a party, the smaller should be the its selection probability.

Probabilistic model

Let us fix some norm $\| \cdot \|$ in $\mathbb{R}^m$.

Intuition: the bigger the is the distance between the opinion of a voter and position of a party, the smaller should be the its selection probability.

Mathematical model:

$$p_i^{(k)}(X) \;\; = \;\; e^{-\|x_k - v_i\|/\mu} / \left[\sum_{q=1}^{\ell} e^{-\|x_q - v_i\|/\mu} \right], \quad k = 1, \ldots, \ell,$$

where $\mu \geq 0$ is the flexibility parameter.

Probabilistic model

Let us fix some norm $\| \cdot \|$ in $\mathbb{R}^m$.

Intuition: the bigger the is the distance between the opinion of a voter and position of a party, the smaller should be the its selection probability.

Mathematical model:

$$p_i^{(k)}(X) \quad = \quad e^{-\|x_k - v_i\|/\mu} / \left[\sum_{q=1}^{\ell} e^{-\|x_q - v_i\|/\mu} \right], \quad k = 1, \dots, \ell,$$

where $\mu \geq 0$ is the flexibility parameter.

Denote by $P_*(X)$ the corresponding voting matrix.

Probabilistic model

Let us fix some norm $\| \cdot \|$ in $\mathbb{R}^m$.

Intuition: the bigger the is the distance between the opinion of a voter and position of a party, the smaller should be the its selection probability.

Mathematical model:

$$p_i^{(k)}(X) \;\; = \;\; e^{-\|x_k - v_i\|/\mu} / \left[\sum_{q=1}^{\ell} e^{-\|x_q - v_i\|/\mu} \right], \quad k = 1, \dots, \ell,$$

where $\mu \geq 0$ is the flexibility parameter.

Denote by $P_*(X)$ the corresponding voting matrix.

NB: $\mu = 0$ corresponds to the deterministic choice of the closest party.

Behavior of political parties

Behavior of political parties

Denote by $c_k \in \mathcal{V}$ the *core value* of kth party.

Behavior of political parties

Denote by $c_k \in \mathcal{V}$ the *core value* of kth party.

For estimating its current position,

Behavior of political parties

Denote by $c_k \in \mathcal{V}$ the *core value* of kth party.

For estimating its current position, define a prox-function $d(x, y)$:

Behavior of political parties

Denote by $c_k \in \mathcal{V}$ the *core value* of kth party.

For estimating its current position, define a prox-function $d(x, y)$:

- $d(x, y) \geq 0$ for all $x, y \in \mathcal{V}$,

Behavior of political parties

Denote by $c_k \in \mathcal{V}$ the *core value* of kth party.

For estimating its current position, define a prox-function $d(x, y)$:

- $d(x, y) \geq 0$ for all $x, y \in \mathcal{V}$, $d(x, x) = 0$.

Behavior of political parties

Denote by $c_k \in \mathcal{V}$ the *core value* of kth party.

For estimating its current position, define a prox-function $d(x,y)$:

- $d(x,y) \geq 0$ for all $x,y \in \mathcal{V}$, $d(x,x) = 0$.
- For each $x \in \mathcal{V}$, $d(x,\cdot)$ is strongly convex in the second argument:

$$d(x,v) \geq d(x,y) + \langle \nabla_2 d(x,y), v - y \rangle + \frac{1}{2}\|v-y\|^2, \quad \forall v,y \in \mathcal{V}.$$

Behavior of political parties

Denote by $c_k \in \mathcal{V}$ the *core value* of kth party.

For estimating its current position, define a prox-function $d(x, y)$:

- $d(x, y) \geq 0$ for all $x, y \in \mathcal{V}$, $d(x, x) = 0$.
- For each $x \in \mathcal{V}$, $d(x, \cdot)$ is strongly convex in the second argument:

$$d(x, v) \geq d(x, y) + \langle \nabla_2 d(x, y), v - y \rangle + \frac{1}{2}\|v - y\|^2, \quad \forall v, y \in \mathcal{V}.$$

Examples:

Behavior of political parties

Denote by $c_k \in \mathcal{V}$ the *core value* of kth party.

For estimating its current position, define a prox-function $d(x, y)$:

- $d(x, y) \geq 0$ for all $x, y \in \mathcal{V}$, $d(x, x) = 0$.
- For each $x \in \mathcal{V}$, $d(x, \cdot)$ is strongly convex in the second argument:

$$d(x, v) \geq d(x, y) + \langle \nabla_2 d(x, y), v - y \rangle + \frac{1}{2}\|v - y\|^2, \quad \forall v, y \in \mathcal{V}.$$

Examples:

- $d_1(x, y) = \eta(y) - \eta(x) - \langle \nabla \eta(x), y - x \rangle = \sum\limits_{i=1}^{\ell} y^{(i)} \ln \frac{y^{(i)}}{x^{(i)}}$, $x, y \in \Delta_\ell$.

Behavior of political parties

Denote by $c_k \in \mathcal{V}$ the *core value* of kth party.

For estimating its current position, define a prox-function $d(x,y)$:

- $d(x,y) \geq 0$ for all $x, y \in \mathcal{V}$, $d(x,x) = 0$.
- For each $x \in \mathcal{V}$, $d(x,\cdot)$ is strongly convex in the second argument:

$$d(x,v) \geq d(x,y) + \langle \nabla_2 d(x,y), v-y \rangle + \tfrac{1}{2}\|v-y\|^2, \quad \forall v, y \in \mathcal{V}.$$

Examples:

- $d_1(x,y) = \eta(y) - \eta(x) - \langle \nabla\eta(x), y-x \rangle = \sum\limits_{i=1}^{\ell} y^{(i)} \ln \frac{y^{(i)}}{x^{(i)}}$, $x, y \in \Delta_\ell$. It is strongly convex in ℓ_1-norm.

Behavior of political parties

Denote by $c_k \in \mathcal{V}$ the *core value* of kth party.

For estimating its current position, define a prox-function $d(x,y)$:

- $d(x,y) \geq 0$ for all $x, y \in \mathcal{V}$, $d(x,x) = 0$.
- For each $x \in \mathcal{V}$, $d(x,\cdot)$ is strongly convex in the second argument:

$$d(x,v) \geq d(x,y) + \langle \nabla_2 d(x,y), v - y \rangle + \tfrac{1}{2}\|v-y\|^2, \quad \forall v, y \in \mathcal{V}.$$

Examples:

- $d_1(x,y) = \eta(y) - \eta(x) - \langle \nabla \eta(x), y - x \rangle = \sum\limits_{i=1}^{\ell} y^{(i)} \ln \frac{y^{(i)}}{x^{(i)}}$,
  $x, y \in \Delta_\ell$. It is strongly convex in ℓ_1-norm.
- $d_2(x,y) \stackrel{\mathrm{def}}{=} \frac{1}{2}\|x-y\|_2^2$, $x, y \in \mathbb{R}^\ell$

Behavior of political parties

Denote by $c_k \in \mathcal{V}$ the *core value* of kth party.

For estimating its current position, define a prox-function $d(x,y)$:

- $d(x,y) \geq 0$ for all $x, y \in \mathcal{V}$, $d(x,x) = 0$.
- For each $x \in \mathcal{V}$, $d(x,\cdot)$ is strongly convex in the second argument:

$$d(x,v) \geq d(x,y) + \langle \nabla_2 d(x,y), v - y \rangle + \tfrac{1}{2}\|v-y\|^2, \quad \forall v, y \in \mathcal{V}.$$

Examples:

- $d_1(x,y) = \eta(y) - \eta(x) - \langle \nabla \eta(x), y - x \rangle = \sum\limits_{i=1}^{\ell} y^{(i)} \ln \frac{y^{(i)}}{x^{(i)}}$, $x, y \in \Delta_\ell$. It is strongly convex in ℓ_1-norm.
- $d_2(x,y) \stackrel{\mathrm{def}}{=} \frac{1}{2}\|x-y\|_2^2$, $x, y \in \mathbb{R}^\ell$ (strongly convex in ℓ_2-norm).

Behavior of political parties

Denote by $c_k \in \mathcal{V}$ the *core value* of kth party.

For estimating its current position, define a prox-function $d(x,y)$:

- $d(x,y) \geq 0$ for all $x,y \in \mathcal{V}$, $d(x,x) = 0$.
- For each $x \in \mathcal{V}$, $d(x,\cdot)$ is strongly convex in the second argument:

$$d(x,v) \geq d(x,y) + \langle \nabla_2 d(x,y), v-y \rangle + \tfrac{1}{2}\|v-y\|^2, \quad \forall v,y \in \mathcal{V}.$$

Examples:

- $d_1(x,y) = \eta(y) - \eta(x) - \langle \nabla\eta(x), y-x \rangle = \sum\limits_{i=1}^{\ell} y^{(i)} \ln \frac{y^{(i)}}{x^{(i)}}$, $x,y \in \Delta_\ell$. It is strongly convex in ℓ_1-norm.
- $d_2(x,y) \stackrel{\mathrm{def}}{=} \frac{1}{2}\|x-y\|_2^2$, $x,y \in \mathbb{R}^\ell$ (strongly convex in ℓ_2-norm).
- Take $\tilde{d}_i(x,y) = d_i(x,y) + \epsilon\|x-y\|_i$ with $\epsilon \geq 0$, $i = 1,2$.

Behavior of political parties

Denote by $c_k \in \mathcal{V}$ the *core value* of kth party.

For estimating its current position, define a prox-function $d(x,y)$:

- $d(x,y) \geq 0$ for all $x, y \in \mathcal{V}$, $d(x,x) = 0$.
- For each $x \in \mathcal{V}$, $d(x,\cdot)$ is strongly convex in the second argument:

$$d(x,v) \geq d(x,y) + \langle \nabla_2 d(x,y), v - y \rangle + \frac{1}{2}\|v-y\|^2, \quad \forall v, y \in \mathcal{V}.$$

Examples:

- $d_1(x,y) = \eta(y) - \eta(x) - \langle \nabla \eta(x), y - x \rangle = \sum\limits_{i=1}^{\ell} y^{(i)} \ln \frac{y^{(i)}}{x^{(i)}}$, $x, y \in \Delta_\ell$. It is strongly convex in ℓ_1-norm.
- $d_2(x,y) \stackrel{\mathrm{def}}{=} \frac{1}{2}\|x-y\|_2^2$, $x, y \in \mathbb{R}^\ell$ (strongly convex in ℓ_2-norm).
- Take $\tilde{d}_i(x,y) = d_i(x,y) + \epsilon\|x-y\|_i$ with $\epsilon \geq 0$, $i = 1, 2$. Additional term gives more chances to keep the core value.

Adjustment to the public opinion

Adjustment to the public opinion

Assumption:

Adjustment to the public opinion

Assumption: For a given voting matrix $P = (p_1, \dots, p_N) \in \Delta_\ell^N$, each political party chooses its current position by minimizing

$$\psi_k(P, x_k) \stackrel{\text{def}}{=} \sum_{i=1}^{N} p_i^{(k)} \|x_k - v_i\| + \frac{1}{\tau} d(c_k, x_k), \; k = 1, \dots, \ell,$$

Adjustment to the public opinion

Assumption: For a given voting matrix $P = (p_1, \ldots, p_N) \in \Delta_\ell^N$, each political party chooses its current position by minimizing

$$\psi_k(P, x_k) \stackrel{\mathrm{def}}{=} \sum_{i=1}^{N} p_i^{(k)} \|x_k - v_i\| + \tfrac{1}{\tau} d(c_k, x_k), \; k = 1, \ldots, \ell,$$

in $x_k \in \mathcal{V}$, where $\tau > 0$ is a *tolerance parameter*.

Adjustment to the public opinion

Assumption: For a given voting matrix $P = (p_1, \dots, p_N) \in \Delta_\ell^N$, each political party chooses its current position by minimizing

$$\psi_k(P, x_k) \stackrel{\mathrm{def}}{=} \sum_{i=1}^{N} p_i^{(k)} \|x_k - v_i\| + \tfrac{1}{\tau} d(c_k, x_k), \ k = 1, \dots, \ell,$$

in $x_k \in \mathcal{V}$, where $\tau > 0$ is a *tolerance parameter*.

The objective function is the expected distance of all attracted voters to the current position of the party, augmented by the proximal term.

Adjustment to the public opinion

Assumption: For a given voting matrix $P = (p_1, \dots, p_N) \in \Delta_\ell^N$, each political party chooses its current position by minimizing

$$\psi_k(P, x_k) \stackrel{\mathrm{def}}{=} \sum_{i=1}^{N} p_i^{(k)} \|x_k - v_i\| + \tfrac{1}{\tau} d(c_k, x_k), \;\; k = 1, \dots, \ell,$$

in $x_k \in \mathcal{V}$, where $\tau > 0$ is a *tolerance parameter*.

The objective function is the expected distance of all attracted voters to the current position of the party, augmented by the proximal term.

Since $\psi_k(P, \cdot)$ is strongly convex, the minimum $x_k^*(P)$ over $\mathcal{V}$ is unique.

Adjustment to the public opinion

Assumption: For a given voting matrix $P = (p_1, \dots, p_N) \in \Delta_\ell^N$, each political party chooses its current position by minimizing

$$\psi_k(P, x_k) \stackrel{\mathrm{def}}{=} \sum_{i=1}^{N} p_i^{(k)} \|x_k - v_i\| + \tfrac{1}{\tau} d(c_k, x_k), \;\; k = 1, \dots, \ell,$$

in $x_k \in \mathcal{V}$, where $\tau > 0$ is a *tolerance parameter*.

The objective function is the expected distance of all attracted voters to the current position of the party, augmented by the proximal term.

Since $\psi_k(P, \cdot)$ is strongly convex, the minimum $x_k^*(P)$ over $\mathcal{V}$ is unique.

Denote $X_*(P) = (x_1^*(P), \dots, x_N^*(P))$.

Consecutive elections

Consecutive elections

Consider now the following *voting process*:

Consecutive elections

Consider now the following *voting process*:

Set $X_0 = (c_1, \ldots, c_\ell)$.

Consecutive elections

Consider now the following *voting process*:

Set $X_0 = (c_1, \ldots, c_\ell)$.
Repeat: $P_{t+1} = P_*(X_t), \; X_{t+1} = X_*(P_{t+1}), \; t \geq 0.$

Consecutive elections

Consider now the following *voting process*:

> **Set** $X_0 = (c_1, \ldots, c_\ell)$.
> **Repeat:** $P_{t+1} = P_*(X_t), \; X_{t+1} = X_*(P_{t+1}), \; t \geq 0$.

We can formally define $P_0 = 0$. Then $X_0 = X_*(P_0)$.

Consecutive elections

Consider now the following *voting process*:

Set $X_0 = (c_1, \dots, c_\ell)$.
Repeat: $P_{t+1} = P_*(X_t), \; X_{t+1} = X_*(P_{t+1}), \; t \geq 0.$

We can formally define $P_0 = 0$. Then $X_0 = X_*(P_0)$.

Interpretation:

Consecutive elections

Consider now the following *voting process*:

> **Set** $X_0 = (c_1, \dots, c_\ell)$.
> **Repeat:** $P_{t+1} = P_*(X_t),\ X_{t+1} = X_*(P_{t+1}),\ t \geq 0$.

We can formally define $P_0 = 0$. Then $X_0 = X_*(P_0)$.

Interpretation: given the current positions of political parties X_t,

Consecutive elections

Consider now the following *voting process*:

> **Set** $X_0 = (c_1, \dots, c_\ell)$.
> **Repeat:** $P_{t+1} = P_*(X_t), \; X_{t+1} = X_*(P_{t+1}), \; t \geq 0$.

We can formally define $P_0 = 0$. Then $X_0 = X_*(P_0)$.

Interpretation: given the current positions of political parties X_t, voters form their preferences P_{t+1} for the next elections.

Consecutive elections

Consider now the following *voting process*:

> **Set** $X_0 = (c_1, \dots, c_\ell)$.
> **Repeat:** $P_{t+1} = P_*(X_t),\ X_{t+1} = X_*(P_{t+1}),\ t \geq 0$.

We can formally define $P_0 = 0$. Then $X_0 = X_*(P_0)$.

Interpretation: given the current positions of political parties X_t, voters form their preferences P_{t+1} for the next elections.

After observing the results, parties form their new positions X_{t+1}.

Consecutive elections

Consider now the following *voting process*:

Set $X_0 = (c_1, \ldots, c_\ell)$.
Repeat: $P_{t+1} = P_*(X_t), \; X_{t+1} = X_*(P_{t+1}), \; t \geq 0$.

We can formally define $P_0 = 0$. Then $X_0 = X_*(P_0)$.

Interpretation: given the current positions of political parties X_t, voters form their preferences P_{t+1} for the next elections.

After observing the results, parties form their new positions X_{t+1}.

Definition:

Consecutive elections

Consider now the following *voting process*:

Set $X_0 = (c_1, \dots, c_\ell)$.
Repeat: $P_{t+1} = P_*(X_t),\ X_{t+1} = X_*(P_{t+1}),\ t \geq 0$.

We can formally define $P_0 = 0$. Then $X_0 = X_*(P_0)$.

Interpretation: given the current positions of political parties X_t, voters form their preferences P_{t+1} for the next elections.

After observing the results, parties form their new positions X_{t+1}.

Definition: Electoral system is called *stable* if the above process has unique limiting point.

Consecutive elections

Consider now the following *voting process*:

$$\textbf{Set } X_0 = (c_1, \ldots, c_\ell).$$
$$\textbf{Repeat: } P_{t+1} = P_*(X_t), \; X_{t+1} = X_*(P_{t+1}), \; t \geq 0.$$

We can formally define $P_0 = 0$. Then $X_0 = X_*(P_0)$.

Interpretation: given the current positions of political parties X_t, voters form their preferences P_{t+1} for the next elections.

After observing the results, parties form their new positions X_{t+1}.

Definition: Electoral system is called *stable* if the above process has unique limiting point.

Recall: we have two positive parameters μ and τ.

Structure of the voting process

Structure of the voting process

Consider a matrix function $F(X) \in \mathbb{R}^{\ell \times N}$ with elements

$$F^{(k,i)}(X) = \|x_k - v_i\|, \quad k = 1, \dots, \ell, \; i = 1, \dots, N.$$

Structure of the voting process

Consider a matrix function $F(X) \in \mathbb{R}^{\ell \times N}$ with elements

$$F^{(k,i)}(X) = \|x_k - v_i\|, \quad k = 1, \dots, \ell,\ i = 1, \dots, N.$$

For $\theta_\mu(s) = -\mu \ln \left(\sum_{k=1}^{\ell} e^{-s^{(k)}/\mu} \right)$,

Structure of the voting process

Consider a matrix function $F(X) \in \mathbb{R}^{\ell \times N}$ with elements

$$F^{(k,i)}(X) = \|x_k - v_i\|, \quad k = 1, \dots, \ell, \; i = 1, \dots, N.$$

For $\theta_\mu(s) = -\mu \ln \left(\sum_{k=1}^{\ell} e^{-s^{(k)}/\mu} \right)$, we have

$$p_i(X) = \nabla \theta_\mu(F(X) e_i), \quad i = 1, \dots, N.$$

Structure of the voting process

Consider a matrix function $F(X) \in \mathbb{R}^{\ell \times N}$ with elements

$$F^{(k,i)}(X) = \|x_k - v_i\|, \quad k = 1, \dots, \ell, \; i = 1, \dots, N.$$

For $\theta_\mu(s) = -\mu \ln \left(\sum\limits_{k=1}^{\ell} e^{-s^{(k)}/\mu} \right)$, we have

$$p_i(X) = \nabla \theta_\mu(F(X) e_i), \quad i = 1, \dots, N.$$

Therefore, $\|p_i(X_{t+1}) - p_i(X_t)\|_1 \leq \frac{1}{\mu} \|F(X_{t+1}) e_i - F(X_t) e_i\|_\infty$.

Structure of the voting process

Consider a matrix function $F(X) \in \mathbb{R}^{\ell \times N}$ with elements

$$F^{(k,i)}(X) = \|x_k - v_i\|, \quad k = 1, \ldots, \ell, \; i = 1, \ldots, N.$$

For $\theta_\mu(s) = -\mu \ln \left(\sum_{k=1}^{\ell} e^{-s^{(k)}/\mu} \right)$, we have

$$p_i(X) = \nabla \theta_\mu(F(X)e_i), \quad i = 1, \ldots, N.$$

Therefore, $\|p_i(X_{t+1}) - p_i(X_t)\|_1 \le \frac{1}{\mu} \|F(X_{t+1})e_i - F(X_t)e_i\|_\infty$.

Norm: $\|P\|_C \stackrel{\mathrm{def}}{=} \max_{1 \le i \le N} \|Pe_i\|_1$.

Structure of the voting process

Consider a matrix function $F(X) \in \mathbb{R}^{\ell \times N}$ with elements

$$F^{(k,i)}(X) = \|x_k - v_i\|, \quad k = 1, \dots, \ell, \; i = 1, \dots, N.$$

For $\theta_\mu(s) = -\mu \ln \left(\sum\limits_{k=1}^{\ell} e^{-s^{(k)}/\mu} \right)$, we have

$$p_i(X) = \nabla \theta_\mu(F(X)e_i), \quad i = 1, \dots, N.$$

Therefore, $\|p_i(X_{t+1}) - p_i(X_t)\|_1 \leq \frac{1}{\mu} \|F(X_{t+1})e_i - F(X_t)e_i\|_\infty$.

Norm: $\|P\|_C \stackrel{\mathrm{def}}{=} \max\limits_{1 \leq i \leq N} \|Pe_i\|_1$. Then $\|P\|_C = 1$, $P \in \Delta_\ell^N$.

Structure of the voting process

Consider a matrix function $F(X) \in \mathbb{R}^{\ell \times N}$ with elements

$$F^{(k,i)}(X) = \|x_k - v_i\|, \quad k = 1, \dots, \ell, \; i = 1, \dots, N.$$

For $\theta_\mu(s) = -\mu \ln \left(\sum\limits_{k=1}^{\ell} e^{-s^{(k)}/\mu} \right)$, we have

$$p_i(X) = \nabla \theta_\mu(F(X)e_i), \quad i = 1, \dots, N.$$

Therefore, $\|p_i(X_{t+1}) - p_i(X_t)\|_1 \leq \frac{1}{\mu} \|F(X_{t+1})e_i - F(X_t)e_i\|_\infty$.

Norm: $\|P\|_C \stackrel{\rm def}{=} \max\limits_{1 \leq i \leq N} \|Pe_i\|_1$. Then $\|P\|_C = 1$, $P \in \Delta_\ell^N$.

Moreover, $\|P_{t+2} - P_{t+1}\|_C \leq \frac{1}{\mu} \|F(X_{t+1}) - F(X_t)\|_\infty, \quad t \geq 0$.

New theorem

New theorem

Th.: Let convex functions $f_i(y)$ be Lipschitz continuous on $\mathcal{V}$:

$$|f_i(y_1) - f_i(y_2)| \leq \|y_1 - y_2\|, \quad \forall y_1, y_2 \in \mathcal{V}, \; i = 1, \ldots, N.$$

New theorem

Th.: Let convex functions $f_i(y)$ be Lipschitz continuous on $\mathcal{V}$:

$$|f_i(y_1) - f_i(y_2)| \leq \|y_1 - y_2\|, \quad \forall y_1, y_2 \in \mathcal{V},\ i = 1, \dots, N.$$

For $q \in \mathbb{R}^N_+$, consider the function

$$\phi(q) \stackrel{\mathrm{def}}{=} \min_{y \in \mathcal{V}} \left\{ \sum_{i=1}^{N} q^{(i)} f_i(y) + d(y) \right\},$$

New theorem

Th.: Let convex functions $f_i(y)$ be Lipschitz continuous on $\mathcal{V}$:

$$|f_i(y_1) - f_i(y_2)| \leq \|y_1 - y_2\|, \quad \forall y_1, y_2 \in \mathcal{V}, \; i = 1, \dots, N.$$

For $q \in \mathbb{R}^N_+$, consider the function

$$\phi(q) \stackrel{\mathrm{def}}{=} \min_{y \in \mathcal{V}} \left\{ \sum_{i=1}^{N} q^{(i)} f_i(y) + d(y) \right\},$$

where $d(y)$ is strongly convex on $\mathcal{V}$ with conv. parameter $\sigma > 0$.

New theorem

Th.: Let convex functions $f_i(y)$ be Lipschitz continuous on $\mathcal{V}$:

$$|f_i(y_1) - f_i(y_2)| \leq \|y_1 - y_2\|, \quad \forall y_1, y_2 \in \mathcal{V}, \; i = 1, \ldots, N.$$

For $q \in \mathbb{R}^N_+$, consider the function

$$\phi(q) \stackrel{\mathrm{def}}{=} \min_{y \in \mathcal{V}} \left\{ \sum_{i=1}^{N} q^{(i)} f_i(y) + d(y) \right\},$$

where $d(y)$ is strongly convex on $\mathcal{V}$ with conv. parameter $\sigma > 0$.

Denote by $y_*(q)$ the optimal solution of the above problem.

New theorem

Th.: Let convex functions $f_i(y)$ be Lipschitz continuous on $\mathcal{V}$:

$$|f_i(y_1) - f_i(y_2)| \leq \|y_1 - y_2\|, \quad \forall y_1, y_2 \in \mathcal{V}, \ i = 1, \dots, N.$$

For $q \in \mathbb{R}^N_+$, consider the function

$$\phi(q) \stackrel{\mathrm{def}}{=} \min_{y \in \mathcal{V}} \left\{ \sum_{i=1}^N q^{(i)} f_i(y) + d(y) \right\},$$

where $d(y)$ is strongly convex on $\mathcal{V}$ with conv. parameter $\sigma > 0$.

Denote by $y_*(q)$ the optimal solution of the above problem.

Then for all $q_1, q_2 \in \mathbb{R}^N_+$ we have

$$\max_{1 \leq i \leq N} |f_i(y_*(q_1)) - f_i(y_*(q_2))| \ \leq \ \|y_*(q_1) - y_*(q_2)\|$$

New theorem

Th.: Let convex functions $f_i(y)$ be Lipschitz continuous on $\mathcal{V}$:

$$|f_i(y_1) - f_i(y_2)| \leq \|y_1 - y_2\|, \quad \forall y_1, y_2 \in \mathcal{V}, \ i = 1, \dots, N.$$

For $q \in \mathbb{R}^N_+$, consider the function

$$\phi(q) \stackrel{\rm def}{=} \min_{y \in \mathcal{V}} \left\{ \sum_{i=1}^{N} q^{(i)} f_i(y) + d(y) \right\},$$

where $d(y)$ is strongly convex on $\mathcal{V}$ with conv. parameter $\sigma > 0$.

Denote by $y_*(q)$ the optimal solution of the above problem.

Then for all $q_1, q_2 \in \mathbb{R}^N_+$ we have

$$\max_{1 \leq i \leq N} |f_i(y_*(q_1)) - f_i(y_*(q_2))| \ \leq \ \|y_*(q_1) - y_*(q_2)\| \leq \frac{1}{\sigma} \|q_1 - q_2\|_1.$$

New theorem

Th.: Let convex functions $f_i(y)$ be Lipschitz continuous on $\mathcal{V}$:

$$|f_i(y_1) - f_i(y_2)| \le \|y_1 - y_2\|, \quad \forall y_1, y_2 \in \mathcal{V},\ i = 1, \dots, N.$$

For $q \in \mathbb{R}^N_+$, consider the function

$$\phi(q) \stackrel{\mathrm{def}}{=} \min_{y \in \mathcal{V}} \left\{ \sum_{i=1}^N q^{(i)} f_i(y) + d(y) \right\},$$

where $d(y)$ is strongly convex on $\mathcal{V}$ with conv. parameter $\sigma > 0$.

Denote by $y_*(q)$ the optimal solution of the above problem.

Then for all $q_1, q_2 \in \mathbb{R}^N_+$ we have

$$\max_{1 \le i \le N} |f_i(y_*(q_1)) - f_i(y_*(q_2))| \ \le\ \|y_*(q_1) - y_*(q_2)\| \le \tfrac{1}{\sigma}\|q_1 - q_2\|_1.$$

Corollary: Since $\nabla \phi(q) = (f_1(y_*(q)), \dots, f_N(y^*(q)))^T$,

New theorem

Th.: Let convex functions $f_i(y)$ be Lipschitz continuous on $\mathcal{V}$:

$$|f_i(y_1) - f_i(y_2)| \leq \|y_1 - y_2\|, \quad \forall y_1, y_2 \in \mathcal{V},\ i = 1, \dots, N.$$

For $q \in \mathbb{R}^N_+$, consider the function

$$\phi(q) \stackrel{\mathrm{def}}{=} \min_{y \in \mathcal{V}} \left\{ \sum_{i=1}^{N} q^{(i)} f_i(y) + d(y) \right\},$$

where $d(y)$ is strongly convex on $\mathcal{V}$ with conv. parameter $\sigma > 0$.

Denote by $y_*(q)$ the optimal solution of the above problem.

Then for all $q_1, q_2 \in \mathbb{R}^N_+$ we have

$$\max_{1 \leq i \leq N} |f_i(y_*(q_1)) - f_i(y_*(q_2))| \ \leq \ \|y_*(q_1) - y_*(q_2)\| \leq \frac{1}{\sigma} \|q_1 - q_2\|_1.$$

Corollary: Since $\nabla \phi(q) = (f_1(y_*(q)), \dots, f_N(y^*(q)))^T$, we have

$$\|\nabla \phi(q_1) - \nabla \phi(q_2)\|_\infty \leq \frac{1}{\sigma} \|q_1 - q_1\|_1.$$

Consequences

Consequences

Define another norm $\|P\|_R = \max_{1 \le k \le \ell} \|P^T e_k\|_1$.

Consequences

Define another norm $\|P\|_R = \max\limits_{1 \le k \le \ell} \|P^T e_k\|_1$.

Since $\sigma = \frac{1}{\tau}$,

Consequences

Define another norm $\|P\|_R = \max_{1 \le k \le \ell} \|P^T e_k\|_1$.

Since $\sigma = \frac{1}{\tau}$, we have

$$\|F(X_{t+1}) - F(X_t)\|_\infty = \|F(X_*(P_{t+1})) - F(X_*(P_t))\|_\infty$$

Consequences

Define another norm $\|P\|_R = \max\limits_{1 \le k \le \ell} \|P^T e_k\|_1$.

Since $\sigma = \frac{1}{\tau}$, we have

$$\begin{aligned} \|F(X_{t+1}) - F(X_t)\|_\infty &= \|F(X_*(P_{t+1})) - F(X_*(P_t))\|_\infty \\ &\le \tau \|P_{t+1} - P_t\|_R, \qquad t \ge 0. \end{aligned}$$

Consequences

Define another norm $\|P\|_R = \max\limits_{1 \le k \le \ell} \|P^T e_k\|_1$.

Since $\sigma = \frac{1}{\tau}$, we have

$$\begin{aligned} \|F(X_{t+1}) - F(X_t)\|_\infty &= \|F(X_*(P_{t+1})) - F(X_*(P_t))\|_\infty \\ &\le \tau \|P_{t+1} - P_t\|_R, \qquad t \ge 0. \end{aligned}$$

Taking into account that $\|P\|_R \le N\|P\|_\infty \le N\|P\|_C$,

Consequences

Define another norm $\|P\|_R = \max\limits_{1 \leq k \leq \ell} \|P^T e_k\|_1$.

Since $\sigma = \frac{1}{\tau}$, we have

$$\begin{aligned} \|F(X_{t+1}) - F(X_t)\|_\infty &= \|F(X_*(P_{t+1})) - F(X_*(P_t))\|_\infty \\ &\leq \tau \|P_{t+1} - P_t\|_R, \qquad t \geq 0. \end{aligned}$$

Taking into account that $\|P\|_R \leq N\|P\|_\infty \leq N\|P\|_C$, we obtain

$$\|P_{t+2} - P_{t+1}\|_C \leq \frac{\tau}{\mu} N \|P_{t+1} - P_t\|_C, \; t \geq 0.$$

Main theorem

Main theorem

Theorem: Let $\tau < \frac{\mu}{N}$.

Main theorem

Theorem: Let $\tau < \frac{\mu}{N}$.

Then the corresponding electoral system is stable.

Main theorem

Theorem: Let $\tau < \frac{\mu}{N}$.

Then the corresponding electoral system is stable.

Moreover, for the stationary voting matrix $P_* = \lim\limits_{t\to\infty} P_t$ we have

$$\|P_t - P_*\|_C \leq \frac{\gamma^t}{1-\gamma}, \quad t \geq 0,$$

where $\gamma = \frac{\tau}{\mu} N < 1$.

Main theorem

Theorem: Let $\tau < \frac{\mu}{N}$.

Then the corresponding electoral system is stable.

Moreover, for the stationary voting matrix $P_* = \lim\limits_{t\to\infty} P_t$ we have

$$\|P_t - P_*\|_C \leq \frac{\gamma^t}{1-\gamma}, \quad t \geq 0,$$

where $\gamma = \frac{\tau}{\mu} N \ < \ 1$.

Auxiliary problems:

Main theorem

Theorem: Let $\tau < \frac{\mu}{N}$.

Then the corresponding electoral system is stable.

Moreover, for the stationary voting matrix $P_* = \lim\limits_{t\to\infty} P_t$ we have

$$\|P_t - P_*\|_C \le \frac{\gamma^t}{1-\gamma}, \quad t \ge 0,$$

where $\gamma = \frac{\tau}{\mu} N \; < \; 1$.

Auxiliary problems:

$$\min_{x_k \in \mathcal{V}} \left\{ \frac{1}{N} \sum_{i=1}^{N} p_i^{(k)} \|x_k - v_i\| + \frac{1}{\gamma\mu} d(c_k, x_k) \right\}, \; k = 1, \dots, \ell.$$

Optimization formulation

Optimization formulation

Problem:

$$\min_{P \in \Delta_\ell^N} \min_{\substack{x_k \in \mathcal{V} \\ k=1,\dots,\ell}} \left\{ \mu \sum_{i=1}^{N} \eta(P^T e_i) + \sum_{\substack{1 \le i \le N \\ 1 \le k \le \ell}} p_i^{(k)} \|x_k - v_i\| + \frac{1}{\tau} \sum_{k=1}^{\ell} d(c_k, x_k) \right\}.$$

Optimization formulation

Problem:

$$\min_{P \in \Delta_\ell^N} \min_{\substack{x_k \in \mathcal{V} \\ k=1,\dots,\ell}} \left\{ \mu \sum_{i=1}^N \eta(P^T e_i) + \sum_{\substack{1 \le i \le N \\ 1 \le k \le \ell}} p_i^{(k)} \|x_k - v_i\| + \frac{1}{\tau} \sum_{k=1}^{\ell} d(c_k, x_k) \right\}.$$

Note:

Optimization formulation

Problem:

$$\min_{P \in \Delta_\ell^N} \min_{\substack{x_k \in \mathcal{V} \\ k=1,\dots,\ell}} \left\{ \mu \sum_{i=1}^N \eta(P^T e_i) + \sum_{\substack{1 \le i \le N \\ 1 \le k \le \ell}} p_i^{(k)} \|x_k - v_i\| + \frac{1}{\tau} \sum_{k=1}^{\ell} d(c_k, x_k) \right\}.$$

Note:

- Our voting process is an *alternating minimization* scheme for this objective.

Optimization formulation

Problem:

$$\min_{P \in \Delta_\ell^N} \min_{\substack{x_k \in \mathcal{V} \\ k=1,\dots,\ell}} \left\{ \mu \sum_{i=1}^{N} \eta(P^T e_i) + \sum_{\substack{1 \le i \le N \\ 1 \le k \le \ell}} p_i^{(k)} \|x_k - v_i\| + \frac{1}{\tau} \sum_{k=1}^{\ell} d(c_k, x_k) \right\}.$$

Note:

- Our voting process is an *alternating minimization* scheme for this objective.
- This objective is *not convex*.

Optimization formulation

Problem:

$$\min_{P \in \Delta_\ell^N} \min_{\substack{x_k \in \mathcal{V} \\ k=1,\dots,\ell}} \left\{ \mu \sum_{i=1}^{N} \eta(P^T e_i) + \sum_{\substack{1 \le i \le N \\ 1 \le k \le \ell}} p_i^{(k)} \| x_k - v_i \| + \frac{1}{\tau} \sum_{k=1}^{\ell} d(c_k, x_k) \right\}.$$

Note:

- Our voting process is an *alternating minimization* scheme for this objective.
- This objective is *not convex*.
- Nevertheless, we prove uniqueness of the solution and global convergence of the process.

Conclusion

Conclusion

1. Our "community detection" model has unique solution.

Conclusion

1. Our "community detection" model has unique solution.

2. This is achieved by probabilistic "fuzzy" separation.

Conclusion

1. Our "community detection" model has unique solution.
2. This is achieved by probabilistic "fuzzy" separation.
3. Other applications: clustering with predefined "centers".

Conclusion

1. Our "community detection" model has unique solution.

2. This is achieved by probabilistic "fuzzy" separation.

3. Other applications: clustering with predefined "centers".

Main question: Can we achieve this in other situations?

Thank you for your attention!