VI Traditional School
Stabilization problem for a scalar equation with one state and two input delays

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Moscow, 2014

Time-delay systems

- System with state delay only: $\dot{X}(t) = X(t-D) + U(t)$ (see [2]).
- System with input delay only: $\dot{X}(t) = X(t) + U(t-D)$ (see [1, 3]).
- System with simultaneous input and state delay: $\dot{X}(t) = X(t-D_1) + U(t-D_2)$ (see [4], [5]).

The area of control design for systems with both input and state delay is underdeveloped [1].

Considered equation

Given a time-delay scalar equation of the form

$$\frac{dx(t)}{dt} = a_0 x(t) + a_1 x(t-h) + b_1 u(t-\tau_1) + b_2 u(t-\tau_2). \quad (1)$$

System delays satisfy the inequalities

$$0 < h < \tau_1 < \tau_2. \quad (2)$$

Problem formulation

We assume that real costants f_0, f_1 are such that the equation

$$\frac{dx(t)}{dt} = (a_0 + f_0)x(t) + (a_1 + f_1)x(t-h) \tag{3}$$

is exponentially stable, i.e.,

$$|x(t,\varphi,\psi)| \leqslant \zeta e^{-\sigma t}||\varphi||_h, \ \ t \geqslant 0. \tag{4}$$

Problem: Find a control law under which equation (1) coincides with (3) for $t \geqslant \tau_2$.

Control law

$$b_1 u(t) + b_2 u(t+\tau_1-\tau_2) = f_0 x(t+\tau_1) + f_1 x(t+\tau_1-h), \quad t \geq \tau_2. \quad (5)$$

Cauchy formula [Bellman R. E., Cooke K. L.]:

$$\begin{aligned} x(t+\tau_1) =& k(\tau_1)x(t) + \int_{-h}^{0} k(\tau_1-\theta-h)a_1 x(t+\theta)d\theta \\ &+ \int_{t-\tau_1}^{t} k(t+\tau_1-\xi)b_1 u(\xi)d\xi \\ &+ \int_{t-\tau_2}^{t+\tau_1-\tau_2} k(t+\tau_2-\xi)b_2 u(\xi)d\xi. \end{aligned}$$

Here $k(t)$ is a fundamental solution of the equation (1).

$$
\begin{aligned}
b_1 u(t) + b_2 u(t+\tau_1-\tau_2) &= (f_0 k(\tau_1) + f_1 k(\tau_1 - h))x(t) \qquad (6) \\
&+ f_0 \int_{-\tau_1}^{0} k(-\varphi) b_1 u(t+\varphi) d\varphi + f_0 \int_{-\tau_2}^{\tau_1-\tau_2} k(\tau_1-\tau_2-\gamma) b_2 u(t+\gamma) d\gamma \\
&+ f_1 \int_{-\tau_1}^{-h} k(-\varphi - h) b_1 u(t+\varphi) d\varphi \\
&+ f_1 \int_{-\tau_2}^{\tau_1-\tau_2-h} k(\tau_1-\tau_2-h-\gamma) b_2 u(t+\gamma) d\gamma \\
&+ f_0 \int_{-h}^{0} k(\tau_1-\theta-h) a_1 x(t+\theta) d\theta + f_1 \int_{-h}^{0} k(\tau_1-\theta-2h) a_1 x(t+\theta) d\theta
\end{aligned}
$$

Equations (1) and (6) form the closed-loop system.

Characteristic function of the closed-loop system

The characteristic function of the closed-loop system is of the form

$$q(s) = \left(b_1 + b_2 e^{-s(\tau_2 - \tau_1)}\right)\left(s - (a_0 + f_0) - (a_1 + f_1)e^{-sh}\right). \quad (7)$$

The condition

$$|b_2| < |b_1| \quad (8)$$

guarantees that the roots of the first factor lie in the open left half of the complex plane. The roots of the second factor lie in the open left half of the complex plane according to the choice of the control coefficients.

Exponential estimate of function $x(t)$

For $t \in [0, \tau_2]$:

$$|x(t, \varphi, \psi)| \leqslant \nu \left(||\varphi||_h + ||\psi||_{\tau_2} \right), \tag{9}$$

where

$$\nu = \max_{t \in [0, \tau_2]} \{\eta(1 + h|a_1|), \eta(\tau_1|b_1| + \tau_2|b_2|)\}, \ \eta = \max_{t \in [0, \tau_2]} |k(t)|.$$

For $t \geqslant \tau_2$:

$$|x(t, \varphi, \psi)| \leqslant \gamma e^{-\sigma t} \left(||\varphi||_h + ||\psi||_{\tau_2} \right), \ \gamma = \nu \zeta e^{\sigma \tau_2}. \tag{10}$$

Exponential estimate of function $u(t)$

$$b_1 u(t) + b_2 u(t+\tau_1-\tau_2) = f_0 x(t+\tau_1) + f_1 x(t+\tau_1-h), \quad t \geq \tau_2.$$

We consider the inhomogeneous equation

$$b_1 u(t) + b_2 u(t-\tau) = m(t). \tag{11}$$

According to (10) $m(t)$ admit an estimate

$$|m(t)| \leqslant \mu e^{-\sigma t}\left(||\varphi||_h + ||\psi||_{\tau_2}\right), \ \mu = \gamma\left(|f_0|e^{-\sigma\tau_1} + |f_1|e^{-\sigma(\tau_1-h)}\right).$$

Estimate of function $u(t)$:

$$|u(t,\varphi,\psi)| \leqslant \lambda e^{-\sigma t}(||\varphi||_h + ||\psi||_{\tau_2}), \ t \geqslant 0. \tag{12}$$

Main result

Theorem 1: If the coefficients b_1, b_2 of the equation (1) satisfy the inequality $|b_2| < |b_1|$, and parameters f_0, f_1 are such that the equation (3) is exponentially stable, then the solutions of the closed-loop system (1), (6) admit the exponential estimate of the form

$$|x(t,\varphi,\psi)| + |u(t,\varphi,\psi)| \leqslant (\gamma+\lambda)e^{-\sigma t}(||\varphi||_h + ||\psi||_{\tau_2}),\ t \geqslant 0.$$

Example

Given the scalar equation

$$\frac{dx(t)}{dt} = x(t-1) + u(t-1{,}5) + 0{,}5u(t-1{,}8). \tag{13}$$

We find a control law under which (13) for $t \geqslant \tau_2$ coincides with the exponential stable equation

$$\frac{dx(t)}{dt} = -x(t). \tag{14}$$

To this end we set

$$f_0 = -1, \; f_1 = -1.$$

Then, the control law is of the form:

$$u(t) + 0{,}5u(t-0{,}3) = 0{,}5\int\limits_{-1{,}8}^{-1{,}3} \gamma u(t+\gamma)d\gamma + \int\limits_{-1{,}5}^{-1} \varphi u(t+\varphi)d\varphi$$

$$+ 0{,}15\int\limits_{-1{,}8}^{-1{,}3} u(t+\gamma)d\gamma - 0{,}5\int\limits_{-1{,}8}^{-0{,}3} u(t+\gamma)d\gamma - \int\limits_{-1{,}5}^{0} u(t+\varphi)d\varphi$$

$$- 2{,}5x(t) + \int\limits_{-1}^{-1{,}5} \theta x(t+\theta)d\theta - 0{,}5\int\limits_{-1}^{-0{,}5} x(t+\theta)d\theta - \int\limits_{-1}^{0} x(t+\theta)d\theta.$$

Equation (13) with the derived control law forms the closed-loop system with the characteristic function

$$q(s) = (1 + 0{,}5e^{-0{,}3s})(s + 1). \qquad (15)$$

All roots of the function $q(s)$ have real part not greater than -1.

Conclusions

- We propose an algorithm for computation of a stabilizing control law for the case of the scalar equation with one state and two input delays.
- The presented control law has the form of an integral equation.
- It is shown that the characteristic function of the closed-loop system consists of two factors.
- We derive conditions under which the roots of both factors lie in the open left half of the complex plane.
- We obtain exponential estimates for the solutions of the closed-loop system.
- The illustrative example is given.

Future plans

- Stabilization of a scalar equation with several state and two input delays.
- Stabilization of a system with several state and input delays.
- Lyapunov-Krasovskii approach for stability analysis of the closed-loop system.

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Thank you for attention!