Modification of Stop-loss Start-gain strategy. Distribution of hedger losses

Vitaly Sobol

Traditional School 2014

1. Settings

Strip (deadband)

$$H \triangleq \{(y,t) : y \in [K, K(1+d)], t \in [0, T]\}. \tag{1}$$

Asset price at the moment t:

$$S(t) \triangleq S_0 - \beta t + \sigma W(t). \tag{2}$$

Minimize mean losses

$$\overline{L}(d,T) \triangleq \mathbf{M}[L(d,T)] \to \min_d . \tag{3}$$

Maximize probability function

$$P_\varphi(d,T) \triangleq \mathcal{P}\{L(d,T) \leq \varphi\} \to \max_d . \tag{4}$$

Minimize quantile function

$$\varphi_\alpha(d,T) \triangleq \min\{\varphi : P_\varphi(d,T) \geq \alpha\} \to \min_d . \tag{5}$$

2. Strip crossing count

Let's denote:
η_T^- — the number of downward crossings of the strip H by $S(t)$ in a time T;
η_T^+ — the number of upward crossings of the strip H by $S(t)$ in a time T;
$\rho^- = -K$ — hedger's losses on one downward crossig, if option was not excercized;
$\rho^+ = K(1+d)(1+\theta)$ — hedger's losses on one upward crossig, if option was not excercized;

Values η_T^- и η_T^+ are random and dependent. For each possible realization of η_T^- value η_T^+ can be equal to η_T^- or $\eta_T^- + 1$.

3. Excercise of option

We denote the probability of excercising of the option by optionholder on one crossing as

$$p(d) = 1 - e^{-\lambda K d}, \lambda > 0.$$

Let $\{\nu_i\}_{i=1}^{\infty}$ be the series of i.i.d Bernoulli random values, which describes the possible excercise of the option at the i-th crossing:

$$\mathcal{P}\{\nu_i = 1\} = p(d), \text{ for each } i = 1, 2, \cdots .$$

The asset price at the moment of early excercise is random and equal to $K + \zeta$, wher ζ is a random value, with support $[0, Kd]$.
The cumulative probability distribution function of ζ let's define as

$$F_\zeta(s) = \begin{cases} 1, & \text{если } s > Kd; \\ \dfrac{1 - e^{-\lambda s}}{1 - e^{-\lambda K d}}, & \text{если } 0 \le s \le Kd; \\ 0, & \text{если } s < 0. \end{cases}$$

Hedger's losses on i-th crossing:

$$l_i \triangleq \begin{cases} \nu_i((K+\zeta)(1+\theta)-K)+(1-\nu_i)\rho^+, & i=1; \\ \left(\prod_{j=1}^{i-1}(1-\nu_j)\right)(\nu_i((K+\zeta)(1+\theta)-K)+(1-\nu_i)\rho^+), & i=2m+1; \\ \left(\prod_{j=1}^{i-1}(1-\nu_j)\right)\rho^-, & i=2m. \end{cases}$$

Possible losses when no crossings happened:

$$l_0 \triangleq \nu_0((K+\zeta)(1+\theta)-K).$$

Total losses of the hedger are equal to

$$L(d,T) \triangleq \begin{cases} \sum_{i=0}^{2m} l_i, & \text{if } \eta^+ + \eta^- = 2m; \\ \sum_{i=0}^{2m+1} l_i - \left(\prod_{j=1}^{2m+1}(1-\nu_j)\right)K, & \text{if } \eta^+ + \eta^- = 2m+1, \end{cases} \tag{6}$$

где $m = 0, 1, 2, \cdots$.

The distributions of random values η_T^- и η_T^+ are defined as follows:

Theorem 1

The number of downward crossings is less than or equal to m with probability

$$\mathcal{P}\{\eta_T^- \geq m\} = \exp\left\{\frac{2m\beta Kd}{\sigma}\right\}[1 - \Phi\left(\frac{\beta}{\sigma}\sqrt{T} + \frac{2mKd}{\sigma\sqrt{T}}\right) + \\ + \exp\left\{-\frac{2m\beta Kd}{\sigma}\right\}\Phi\left(\frac{\beta}{\sigma}\sqrt{T} - \frac{2mKd}{\sigma\sqrt{T}}\right)].$$

The number of upward crossings is less than or equal m with probability

$$\mathcal{P}\{\eta_T^+ \geq m\} = \exp\left\{-\frac{2m\beta Kd}{\sigma}\right\}[1 - \Phi\left(-\frac{\beta}{\sigma}\sqrt{T} + \frac{(2m-1)Kd}{\sigma\sqrt{T}}\right) + \\ + \exp\left\{\frac{(2m-1)\beta Kd}{\sigma}\right\}\Phi\left(-\frac{\beta}{\sigma}\sqrt{T} - \frac{(2m-1)Kd}{\sigma\sqrt{T}}\right)].$$

6. Hedger mean losses

Let's write the mean losses of the hedger's as total losses expectation:

$$\overline{L}(d,T) = \sum_{i=0}^{\infty} \mathbf{M}\left[L(d,T)|\eta^{+} + \eta^{-} = i\right] \mathcal{P}\{\eta^{+} + \eta^{-} = i\}.$$

Considering, that option could remain "in-money" and not exercised before T, we obtain the value of mean losses

$$\overline{L}(d,T) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \mathbf{M}[l_j]\mathcal{P}\{\eta^{+} + \eta^{-} = k\} -$$

$$- \sum_{k=0}^{\infty} (1 - p(d))^{2k+1} K \mathcal{P}\{\eta^{+} + \eta^{-} = 2k+1\}. \quad (7)$$

7. Conditional distribution of losses

Let's define the conditional distribution function of the losses

$$P_\varphi(d,T,k) \triangleq \mathcal{P}\{L(d,T) \le \varphi | \eta^+ + \eta^- = k\}, \text{ for each } \varphi > 0.$$

term "conditional" means known number of strip crossings. Using total probability formula, we obtain unconditional distribution function

$$P_\varphi(d,T) = \sum_{k=0}^{\infty} P_\varphi(d,T,k) \cdot \mathcal{P}\{\eta^+ + \eta^- = k\}.$$

Let us define conditional and unconditional quantile functions as

$$\varphi_\alpha(d,T,k) \triangleq \min\{\varphi : P_\varphi(d,T,k) \ge \alpha\}, \qquad (8)$$

$$\varphi_\alpha(d,T) \triangleq \min\{\varphi : P_\varphi(d,T) \ge \alpha\}. \qquad (9)$$

8. Conditional distribution of losses

Let's denote

$$k^*(\varphi) = \left[\frac{\varphi}{\rho^+ - \rho^-}\right], \varphi^* = \frac{\varphi - 2k^*(\varphi)(\rho^+ + \rho^-) + K}{1+\theta}$$

(i) If $i < 2k^*(\varphi)$: $P_\varphi(d, T, k) = 1$.
(ii) If $i = 2k^*(\varphi)$:

$$P_\varphi(d, T, k) = 1-(1-p(d))^{2k^*(\varphi)}+(1-p(d))^{2k^*(\varphi)}\frac{\mathcal{P}\{K \le S(T) \le \varphi^*\}}{\mathcal{P}\{S(T) \le K(1+d)\}}.$$

(iii) If $i = 2k^*(\varphi) + 1$:

$$P_\varphi(d, T, k) = (1-p(d))^{2k^*(\varphi)}p(d)\mathcal{P}\{\zeta \le \varphi^* - K\}+$$
$$+1-(1-p(d))^{2k^*(\varphi)}+(1-p(d))^{2k^*(\varphi)+1}\frac{\mathcal{P}\{K \le S(T) \le \varphi^*\}}{\mathcal{P}\{K \le S(T)\}}.$$

(iv) if $i > 2k^*(\varphi) + 1$:

$$P_\varphi(d, T, k) = 1-(1-p(d))^{2k^*(\varphi)}+(1-p(d))^{2k^*(\varphi)}p(d)\mathcal{P}\{\zeta \le \varphi^* - K\}.$$

It's easily seen, that values $\varphi_\alpha(d, T, k)$ are increasing with k and

$$P_{\varphi_\alpha(d,T,0)}(d, T) = \sum_{k=0}^{\infty} P_{\varphi_\alpha(d,T,0)}(d, T, k) \cdot \mathcal{P}\{\eta^+ + \eta^- = k\} \leq$$

$$\leq \sum_{k=0}^{\infty} \alpha \cdot \mathcal{P}\{\eta^+ + \eta^- = k\} = \alpha.$$

Therefore, exists $m > 0$ that:

$$P_{\varphi_\alpha(d,T,m)}(d, T) \geq \alpha$$
$$P_{\varphi_\alpha(d,T,m-1)}(d, T) \leq \alpha.$$

Values $\varphi_\alpha(d, T, m-1)$ and $\varphi_\alpha(d, T, m)$ can be used as upper and lower estimates for unknown unconditional quantile $\varphi_\alpha(d, T)$.

10. Adjustment

Let's assume, that we know the exact value or upper estimate for Lipschitz constant L for function $P_\varphi(d, T)$ on interval $[\varphi_\alpha(d, T, m-1); \varphi_\alpha(d, T, m)]$, now, we can easily construct upper and lower envelopes for function $P_\varphi(d, T)$

$$F^+(\varphi, d, T) = \min\{P_{\varphi_\alpha(d,T,m)}(d, T), P_{\varphi_\alpha(d,T,m-1)}(d, T) + L(\varphi - \varphi_\alpha(d, T, m-1))\},$$

$$F^-(\varphi, d, T) = \min\{P_{\varphi_\alpha(d,T,m-1)}(d, T), P_{\varphi_\alpha(d,T,m)}(d, T) + L(\varphi - \varphi_\alpha(d, T, m))\}.$$

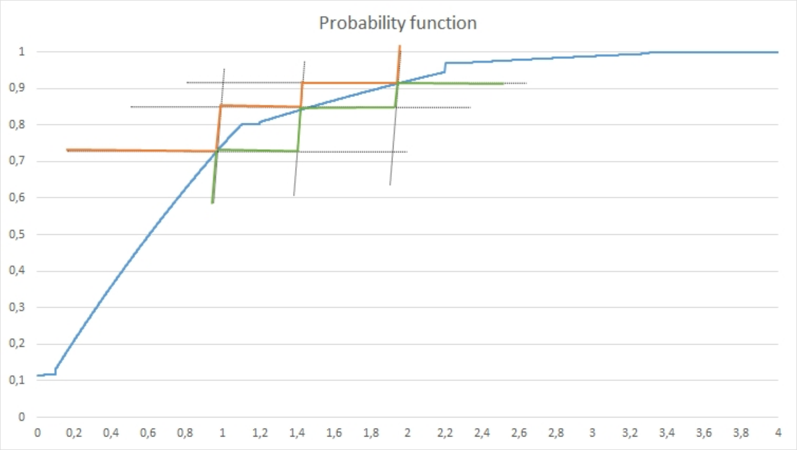
We obtain amended estimates of unconditional quantile:

$$\varphi_\alpha^+(d, T) = \frac{\alpha - P_{\varphi_\alpha(d,T,m)}(d, T)}{L} + \varphi_\alpha(d, T, m),$$

$$\varphi_\alpha^-(d, T) = \frac{\alpha - P_{\varphi_\alpha(d,T,m+1)}(d, T)}{L} + \varphi_\alpha(d, T, m+1).$$

This approach can be used together with dichotomy method for consequent adjustment of estimates of $\varphi_\alpha^+(d, T)$.

11. Adjustment



Considering that for calculation of $P_\varphi(d, T)$ one needs to calculate the sum of series, estimation of Lipschitz constant L can be very time-consuming. Let's assume that we can calculate or estimate constants L_k for conditional distribution functions $P_\varphi(d, T, k)$, then

$$|P_{x_1}(d, T) - P_{x_q}(d, T)| = |\sum_{k=0}^{\infty} P_{x_1}(d, T, k)p_i - \sum_{i=0}^{\infty} P_{x_2}(d, T, k)p_k| = |\sum_{k=0}^{\infty}(P_{x_1}(d, T, k$$

$$\leq \sum_{k=0}^{\infty} |P_{x_1}(d, T, k) - P_{x_2}(d, T, k)|p_k \leq \sum_{k=0}^{\infty} L_k|x_1 - x_2|p_k = |x_1 - x_2| \sum_{k=0}^{\infty} L_k p_k,$$

where $p_k \triangleq \mathcal{P}\{\eta^+ + \eta^- = k\}$. So, we obtain upper estimate for the Lipschitz constant L:

$$L \leq \sum_{k=0}^{\infty} L_k p_k. \qquad (10)$$