Passification based synchronization of cascade Lurie systems with quantized signals

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Problem statement

Consider two dynamical systems of the Lurie type with nonlinear input cascade

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\varphi(y_1) + f_1(t),\ y_1(t) = Cx(t), \qquad (1) \\ \dot{z}(t) &= Az(t) + B\varphi(y_2) + Bu(t) + f_2(t),\ y_2(t) = Cz(t), \qquad (2) \\ \dot{u}(t) &= \psi(u,t) + w(t), \qquad (3) \end{aligned}$$

where $x(t), z(t)$ are the n-dimensional vectors of the object state. $y_1(t), y_2(t)$ are scalar outputs. A is the $n \times n$ matrix, B is the $n \times 1$ matrix, C is the $1 \times n$ matrix, $\varphi(y)$, $\psi(u,t)$ are the continuous non-linearities lying in the sector, $f_i(t)$ – disturbance, $\|f_i(t)\| \leqslant \Delta_{f_i}$. The system (1) will be called the master one, the system (2) will be called the slave one.

Problem statement

Consider quantized feedback control law

$$\tilde{w}(t) = q_\mu(w(t)).$$

Our goal is to synchronize two systems (1),(2) with integrator (3), to achieve zero asymptotic state error: $y_1(t) - y_2(t) \to 0$ where $t \to \infty, i = 1, ..., n.$

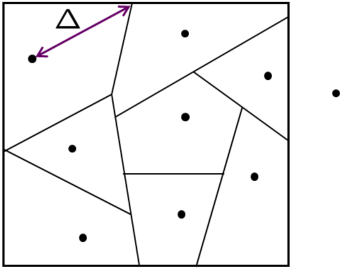
Problem statement

Quantizer [5]:

$$q \colon \mathbb{R}^l \to Q,$$

where Q is a finite subset of $\mathbb{R}^l$. We assume there exist $M > \Delta > 0$:

$$\|y\| \leq M \Rightarrow \|q(y) - y\| \leq \Delta.$$



Dynamic quantizer:

$$q_\mu(y) = \mu q\left(\frac{y}{\mu}\right)$$

with a zooming variable $\mu > 0$.

Evaluation of control

The synchronization error $e(t) = x(t) - z(t)$, the synchronization error in the output $\varepsilon(t) = y_1(t) - y_2(t) = Ce(t)$. Also we assume that $f_i(t) = 0$.

$$\begin{aligned} \dot{e}(t) &= Ae(t) + B\xi(\varepsilon(t), t) - Bu(t),\ \varepsilon(t) = Ce(t) \qquad (4) \\ \dot{u}(t) &= \psi(u, t) + w(t) \qquad (5) \end{aligned}$$

where $\xi(\varepsilon, t) = \varphi(y_1) - \varphi(y_2)$ is a new nonlinearity. The control aim will take the form $\lim_{t\to\infty} e(t) = 0$.
For synthesis of the control $w(t)$ we will resort to the method of backstepping [3].

$$\begin{aligned} \dot{e}(t) &= Ae(t) + B\xi(\varepsilon, t) - Bu(t),\ \varepsilon(t) = Ce(t) \qquad (6) \\ \dot{u}(t) &= KCAe(t) + KCB\xi(\varepsilon, t) + \psi(u, t) + v(t) \qquad (7) \end{aligned}$$

where $v(t) = (-\gamma - KCB)u + \gamma K\varepsilon$.

Conditions of Passification and Asymptotic Stabilization

To obtain the conditions for achieving the aim, we will make the following assumptions:

1. let the linear system $\dot{e}(t) = Ae(t) - Bu(t),\ \sigma(t) = Ce(t)$ be the hyperminimum-phase one;
2. $\xi(\sigma, t)$ lies in the sector, i.e., $a\sigma^2 \leqslant \xi(\sigma, t)\sigma \leqslant b\sigma^2$;
3. $\psi(u, t)$ also lies in the sector, i.e., $cu^2 \leqslant \psi(u, t)u \leqslant du^2$;
4. from the hyperminimum-phase property and the passification theorem [1] it follows that the minimum distance η_0 between the roots of the numerator of a transfer function and the imaginary axis will be positive. We will select the parameters η and K in such a way that $0 < \eta < \eta_0$, $2\|\tilde{D}\|\|P\|\|C\| \max(|a|, |b|) + 2\|P\| \max(|c|, |d|) < \eta\lambda_{min}$, where $\tilde{D} = \begin{pmatrix} B \\ KCB \end{pmatrix}$, P is the positive definite matrix in the Lyapunov quadratic function $V(x) = x^T Px$, λ_{min} is the least eigenvalue of the given matrix.

Conditions of Passification and Asymptotic Stabilization

Theorem

Let the assumptions (1)-(4) be fulfilled. Then there exist numbers K, γ, such that the system (6),(7) will be passive with the quadratic storage function, while the closed system with the control $v(t)$ will be asymptotically stable.

Exponential Synchronization of System with Discrete Controller

Consider the system (6), (7) with the discrete controller
$v(t) = (-\gamma - KCB)u(t_k) + \gamma K \varepsilon(t_k), \;\; t_k \leqslant t \leqslant t_{k+1}$, where $t_k = kh$ are the instants of time with the discretization step h.
The initial system (6), (7) can be represented in the form

$$\dot{x}(t) = \tilde{A}x(t) + \tilde{B}v(t) + \tilde{B}\psi(u,t) + \tilde{D}\xi(\varepsilon,t),$$
$$\tilde{\varepsilon}(t) = \tilde{C}x(t), \;\; v(t) = \tilde{K}\tilde{\varepsilon}(t_k). \quad (8)$$

Let's formulate the theorem of estimation of discretization step for the system with discrete controller by [6].

Exponential Synchronization of System with Discrete Controller

Theorem

Consider the system (8) with the discrete-time controller. Select the discretization step so that the following inequality be fulfilled:

$$\|\tilde{C}\|\varkappa\|\tilde{K}\|e^{L_G h} - \|\tilde{C}\|\varkappa\|\tilde{K}\| \leqslant \min\{L_G - L_G e^{-\eta h}, 2\} \tag{9}$$

$\varkappa$ is the coefficient of the estimate of the system output in terms of the Lyapunov function: $|\tilde{\sigma}| \leqslant \varkappa\sqrt{V}$, L_G is the Lipschitz constant of the right hand side of the system (8). Then the system under consideration is exponentially stable i.e. the synchronization error exponentially tends to zero [6, 7].

The Discrete Controller and disturbances

Consider the system (6), (7) with bounded disturbance $\|f(t)\| \leqslant \Delta_f$ and with the discrete-time controller
$v(t) = (-\gamma - KCB)u(t_k) + \gamma K\varepsilon(t_k), \;\; t_k \leqslant t \leqslant t_{k+1}$, where $t_k = kh$ are the instants of time with the discretization step h.
The initial system (6), (7) can be represented in the form

$$\dot{x}(t) = \tilde{A}x(t) + \tilde{B}v(t) + \tilde{B}\psi(u,t) + \tilde{D}\xi(\varepsilon,t) + f(t),$$
$$\tilde{\varepsilon}(t) = \tilde{C}x(t), \;\; v(t) = \tilde{K}\tilde{\varepsilon}(t_k), \quad (10)$$

where $f(t) = \begin{pmatrix} f_1(t) - f_2(t) \\ 0 \end{pmatrix}$ – bounded disturbance,
$x = \begin{pmatrix} e \\ u \end{pmatrix}$, $\tilde{\varepsilon} = \begin{pmatrix} \varepsilon \\ u \end{pmatrix}$.

The Discrete Controller and disturbances

Theorem

Consider the system (10) with the discrete-time controller and disturbance $f(t)$. Select the discretization step so that the following inequality be fulfilled:

$$L_G e^{-\eta h} + \|\tilde{C}\| \varkappa \|\tilde{K}\| e^{L_G h} - \|\tilde{C}\| \varkappa \|\tilde{K}\| < 0 \qquad (11)$$

$\varkappa$ is the coefficient of the estimate of the system output in terms of the Lyapunov function: $|\tilde{\varepsilon}| \leqslant \varkappa\sqrt{V}$, L_G is the Lipschitz constant of the right side of the system (10). Then $\lim_{t\to\infty} \|x(t)\| \leqslant C_x \Delta_f$, where $C_x = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\frac{1}{\eta}$.

The Discrete Controller and quantizer

Consider the system (6), (7) with the discrete-time controller and quantizer $v(t) = Q((-\gamma - KCB)u(t_k) + \gamma K\varepsilon(t_k)), \;\; t_k \leqslant t \leqslant t_{k+1}$, where $t_k = kh$ are the instants of time with the discretization step h.
The initial system (6), (7) can be represented in the form

$$\dot{x}(t) = \tilde{A}x(t) + \tilde{B}v(t) + \tilde{B}\psi(u,t) + \tilde{D}\xi(\varepsilon,t),$$
$$\tilde{\varepsilon}(t) = \tilde{C}x(t), \; v(t) = \tilde{K}Q(\tilde{\varepsilon}(t_k)) = \tilde{K}\tilde{\varepsilon}(t_k) + \Delta, \quad (12)$$

where $x = \begin{pmatrix} e \\ u \end{pmatrix} \; \tilde{\varepsilon} = \begin{pmatrix} \varepsilon \\ u \end{pmatrix}$.

The Discrete Controller and quantizer

Theorem

Consider the system (8) with the discrete-time controller and quantizer Q. Select the discretization step so that the following inequality be fulfilled:

$$(L_G + \Delta)e^{-\eta h} + \|\tilde{C}\|\varkappa\|\tilde{K}\|e^{(L_G+\Delta)h} - \|\tilde{C}\|\varkappa\|\tilde{K}\| < 0 \qquad (13)$$

$\varkappa$ is the coefficient of the estimate of the system output in terms of the Lyapunov function: $|\tilde{\varepsilon}| \leqslant \varkappa\sqrt{V}$, L_G is the Lipschitz constant of the right side of the system (8). Then $\lim_{t\to\infty} \|x(t)\| \leqslant C_x\Delta$, where $C_x = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\frac{1}{\eta}$.

Control of Nonlinear Lurie Systems with Dynamic Quantizer

Let $q_\mu(z) = \mu q(\frac{z}{\mu})$ dynamic quantizer with the zooming parameter $\mu > 0$.
Consider the system (6), (7) with the controller
$v(t) = q_\mu((-\gamma - KCB)u(t) + \gamma K\varepsilon(t))$.
The initial system (6), (7) can be represented in the form

$$\dot{x}(t) = \tilde{A}x(t) + \tilde{B}v(t) + \tilde{B}\psi(u,t) + \tilde{D}\xi(\varepsilon,t),$$
$$\tilde{\varepsilon}(t) = \tilde{C}x(t),\ v(t) = \tilde{K}q_\mu(\tilde{\varepsilon}(t_k)), \quad (14)$$

where $x = \begin{pmatrix} e \\ u \end{pmatrix}$ $\tilde{\varepsilon} = \begin{pmatrix} \varepsilon \\ u \end{pmatrix}$.

Control of Nonlinear Lurie Systems with Dynamic Quantizer

Assume for the moment that μ is a fixed positive number.

Lemma

Fix an arbitrary ϵ and assume that M is large enough compared to Δ so that we have

$$\sqrt{\lambda_{min}(P)}M > \sqrt{\lambda_{max}(P)}\Theta_x\Delta(1+\epsilon), \tag{15}$$

where $\Theta_x = \frac{2\|PBKC\|}{\chi} > 0$,
$\chi = \eta - \|D\|\|C\| \max(|a|,|b|) - \|B\| \max(|c|,|d|)$ *(see assumptions (1)-(4)).*

Lemma

Then the ellipsoids

$$R_1(\mu) = \{x : x^T P x \leqslant \lambda_{min}(P) M^2 \mu^2\} \tag{16}$$

and

$$R_2(\mu) = \{x : x^T P x \leqslant \lambda_{max}(P) \Theta_x^2 \Delta^2 (1+\epsilon)^2 \mu^2\} \tag{17}$$

are invariant regions for system (14). Moreover, all so- lutions of (14) that start in the ellipsoid (16) enter the smaller ellipsoid (17) in finite time.

Control of Nonlinear Lurie Systems with Dynamic Quantizer

Theorem

Assume that M is large enough compared to Δ so that we have

$$\frac{M}{\Delta} > 2\max\{1, \frac{\|PBCK\|}{\chi}\}\sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}, \tag{18}$$

where $\chi = \eta - \|D\|\|C\| \max(|a|, |b|) - \|B\| \max(|c|, |d|)$.
Then there exists a hybrid quantized feedback control algorithm that makes system (14) globally asymptotically stable.

EXAMPLE

We demonstrate our theorems by the example of a model nonlinear system consisting of two mobile three-wheeled robots in master-slave configuration. The model can be represented in the following way [4]:

$$\begin{aligned} \dot{x}_1(t) &= v\cos(\varphi_1(t)), & \dot{y}_1(t) &= v\cos(\varphi_2(t)), \\ \dot{x}_2(t) &= v\sin(\varphi_1(t)), & \dot{y}_2(t) &= v\sin(\varphi_2(t)), \\ \dot{\varphi}_1(t) &= \omega, & \dot{\varphi}_2(t) &= u(t), \end{aligned} \tag{19}$$

where $u(t), r(t)$ are the control functions, ω is the fixed angular velocity, v is the fixed linear velocity.

EXAMPLE

Represent system (19) in the following way:

$$\begin{aligned} \dot{X}(t) &= AX(t) + Bu(t) + D\xi(t, \sigma(t)), \\ u(t) &= \tilde{K}X(t), \end{aligned} \tag{20}$$

where $X = \begin{bmatrix} e \\ \varepsilon \end{bmatrix}$, $A = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $D = \begin{bmatrix} v \\ 0 \end{bmatrix}$,
$\tilde{K} = \begin{bmatrix} \gamma K & Kv - \gamma \end{bmatrix}$.

EXAMPLE

Consider the Lyapunov quadratic functions $V(X) = X^T PX$. For applying Theorem 3, calculate the parameter $\eta < 0$:

$$\begin{aligned} \dot{V} &= ((A + B\tilde{K})X + D\xi(\varepsilon, t))^T PX + \\ &+ X^T P((A + B\tilde{K})X + D\xi(\varepsilon, t)) \leqslant \\ &\leqslant 2(\|(A + B\tilde{K})\| + \max(|0|, |-2|)v)V \\ &= -\eta V. \end{aligned} \tag{21}$$

Define the vehicle motion velocity $v = 0.1$ m/s. Then

$$\eta = -2(\max(-\gamma, vK) + 2v) > 0. \tag{22}$$

Select γ, K such that (22) holds: $K = -5, \gamma = 0.6$, then $\eta = 0.6$. By Matlab we evaluate the norm of matrix P, $\|P\| = 60.9306$.

EXAMPLE

Let's consider our system with quantizer with quantization error $\Delta = 0.1$

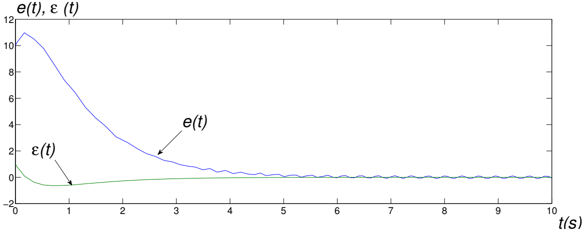


Figure: Errors e_i and ε_i, $i = 1, 2$ in system (19) with quantization error $\Delta = 0.1$.

EXAMPLE

Consider dynamic quantizer. For applying Theorem 5, we calculate χ:

$$\chi = \eta - 2v \tag{23}$$

So for this system we can rewrite inequlity (18) as

$$M > 2567.67\Delta. \tag{24}$$

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Strictly Passive Systems

Basic definitions:

- For a given $g \in \mathbb{R}^l$ a linear time-invariant (LTI) system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \quad x \in \mathbb{R}^n, y \in \mathbb{R}^l, u \in \mathbb{R}$$

  is called ***strictly passive*** if there exist continuous nonnegative functions $V(x)$, $\varphi(x)$ such that $\varphi(x) > 0$ when $x \neq 0$ and for any solution (x, u)

$$V(x(t)) \leq V(x(0)) + \int_0^t \left[y^T(s) g u(s) - \varphi(x(s)) \right] ds.$$

- If $u = Ky + v$ makes LTI system strictly passive with respect to v, LTI system is called ***strictly passifiable***.
- The transfer function $g^T W(s) = g^T C(sI - A)^{-1} B$ is called ***hyper-minimum-phase*** (HMP) if $g^T W(s) \det(sI - A)$ is stable and $g^T CB > 0$.

Strictly Passive Systems

Passification lemma [2]. The following statements are equivalent:

- LTI system is strictly passifiable.
- $g^T W(s) = g^T C(sI - A)^{-1} B$ is HMP.
- $\exists P, \varkappa_*$:

$$P > 0, \quad PA_\varkappa + A_\varkappa^T P < 0, \quad PB = C^T g,$$

where $A_\varkappa = A - B\varkappa_* g^T C$.