Applying invariant ellipsoid approach to system state tracking problem

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Introduction

The aim of the talk is a research on one of possible statements of tracking problem, one of which can be traced back to R. Kalman.

The talk is devoted to system state tracking problem in linear control system. Reference signal is consider satisfying a differential equation, which includes bounded disturbance as one component. Such constraint allows considering broad class of reference signal. The control aim is to make system state as "close" (in some sense) as possible to reference signal.

Suggested approach to problem solution is based on invariant ellipsoid technique; Linear Matrix Inequalities (LMI) approach is used as technical means. Such approach enables us to reformulate the initial problem in a convenient way for solving: searching the minimal bounding ellipsoid containing systems output. Trace criteria is chosen as cost function, which corresponds to minimization of sum of ellipsoid semi axis squares.

From technical perspective problem is reduced to Semi-Definite Programming (SDP) and one-dimensional optimization. Such problems can be effectively solved computationally, for instance, by means of freeware SDPT3 and YALMIP packages for Matlab.

Invariant ellipsoid technique

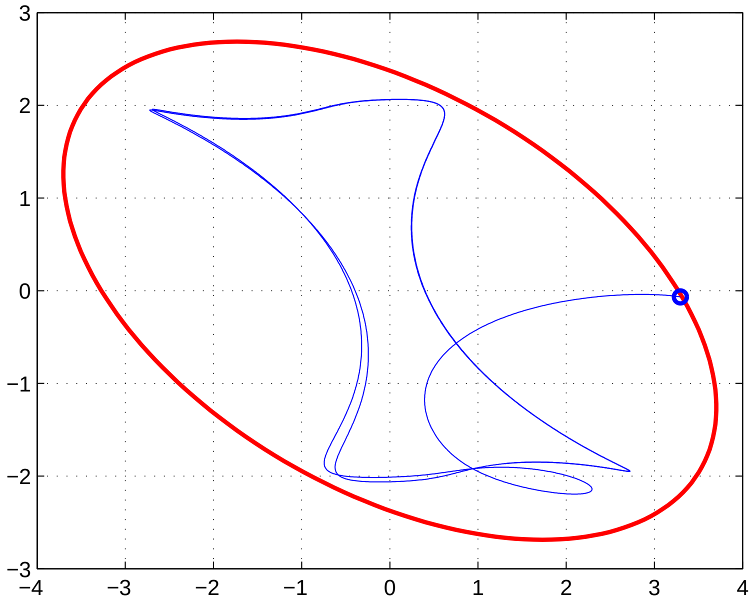
- Ellipsoid with centre in zero of coordinate system

$$\mathcal{E}_x = \{x \in \mathbb{R}^n : \quad x^\top P^{-1} x \le 1\}, \quad P \succ 0, \quad (1)$$

is called invariant for considering system, if

$$x(0) \in \mathcal{E}_x \implies x(t) \in \mathcal{E}_x \qquad \forall t \ge 0$$

i. e. starting at any point of $\mathcal{E}_x$ system trajectory is guaranteed to stay inside $\mathcal{E}_x$ for all feasible disturbances.



- The problem of finding optimal bounding ellipsoid for above mentioned system was solved in [Boyd et al., 1994].

Boyd S., El Ghaoui L., Feron E., Balakrishnan V. Linear Matrix Inequalities in System and Control Theory. SIAM. Philadelphia. 1994.

Synthesis problem

Consider linear continuous control system

$$\dot{x} = Ax + Bu + Df, \quad x(0) = x_0, \tag{2}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times p}, D \in \mathbb{R}^{n\times n}$ — known matrices

$x(t) \in \mathbb{R}^n$ — system state,

$u \in \mathbb{R}^p$ — control

$f(t) \in \mathbb{R}^n$ — reference signal, satisfying following constraints

$$\left\| \begin{matrix} f \\ \dot{f} \end{matrix} \right\| \le 1, \quad \forall t \ge 0.$$

The problem is to find control, which optimally minimize (in below mentioned sense) error $z = x - f \in \mathbb{R}^n$. Then

$$\dot{x} = \dot{f} + \dot{z} = A(f + z) + Bu + Df,$$

from which

$$\dot{z} = Az + Bu + (A + D)f - \dot{f}. \tag{3}$$

Suggested approach

Let us introduce composite vector

$$g = \begin{pmatrix} f \\ \dot{f} \end{pmatrix},$$

where $g \in \mathbb{R}^{2n}$.

System (3) can be rewritten in terms of new state vector z in matrix form in a following way:

$$\dot{z} = Az + Bu + \underbrace{(A + D, -I)}_{D_1} g \tag{4}$$

Let us find regulator in following form

$$u = K_1 z + K_2 g, \tag{5}$$

where $K_1 \in \mathbb{R}^{p \times n}$, $K_2 \in \mathbb{R}^{p \times 2n}$.

Closed-loop system (4) with (5) can be written as

$$\dot{z} = \underbrace{(BK_1 + A)}_{A_c} z + \underbrace{(D_1 + BK_2)}_{D_c} g \tag{6}$$

in terms of new state vector z with disturbance g, which satisfies the following constraint

$$\|g(t)\| \le 1, \quad \forall t \ge 0. \tag{7}$$

Theorem 1

Solution $\widehat{P}, \widehat{Y}, \widehat{K}_2$ of minimization problem

$$\text{tr } P \to \min \tag{8}$$

under constraints

$$\begin{pmatrix} AP + PA^T + \alpha P + BY + Y^T B^T & D_1 + BK_2 \\ (D_1 + BK_2)^T & -\alpha I \end{pmatrix} \leq 0, \qquad P \geq 0, \tag{9}$$

where minimization is made on matrix variables

$$P = P^T \in \mathbb{R}^{n \times n}, \;\; Y \in \mathbb{R}^{p \times n}, \;\; K_2 \in \mathbb{R}^{n \times n}$$

and numeric parameter $\alpha > 0$, defines matrix $\widehat{P}$ of invariant ellipsoid for system (2) static state regulator

$$\widehat{K} = \begin{pmatrix} \widehat{Y}\widehat{P}^{-1} & \widehat{K}_2 \end{pmatrix},$$

dampen external disturbances.

Observations

- Minimization problem (8) under constraints (9) is reduced to Semi-Definite Programming (SDP) problem and one-dimensional optimization by scalar parameter α

- The boundaries of changing interval of parameter α are not known beforehand. To some extent it can be compensated by the fact that target function is convex in all test examples.

- If system (6) has a linear output $y = Cx$, one can consider linear image of ellipsoid and easily modify the theorem.

Example: AC5 from COMPleib

Let us demonstrate the method effectiveness on example of AC5 from Compleib [F. Leibfritz, 2004]:

$$\dot{x} = Ax + Bu + Df,$$

where

$$A = \begin{pmatrix} 0.9081 & 0.0003 & -0.0980 & 0.0038 \\ -0.3868 & 0.0971 & 0.0471 & -0.0008 \\ 0.1591 & -0.0015 & 0.9621 & 0.0003 \\ -0.0198 & 0.0958 & 0.0021 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} -0.0001 & 0.0058 \\ 0.0296 & 0.0153 \\ 0.0012 & -0.0908 \\ 0.0015 & 0.0008 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Let us demand additional constraint on control

$$\|u(t)\| \leq \mu. \tag{10}$$

F. Leibfritz COMPleib: COnstraint Matrix-optimization Problem library - a collection of test examples for nonlinear semidefinite programs, control system design and related problems. Tech.-Report 2004.

Lemma 1

The fulfilment of the constraint

$$\|u(t)\| \leq \mu \quad \forall t \geq 0 \tag{11}$$

for system (2) and regulator (5) is guarenteed by fulfilment of condition

$$\begin{pmatrix} \mu^2 I & Y & K_2 \\ Y^T & \varepsilon P & 0 \\ K_2^T & 0 & (1-\varepsilon) I \end{pmatrix} \geq 0,$$

where

$$P = P^T \in \mathbb{R}^{n \times n}, \quad Y \in \mathbb{R}^{p \times n}, \quad K_2 \in \mathbb{R}^{n \times n},$$

with some

$$0 < \varepsilon < 1.$$

Example: AC5

In accordance with lemma 1 and theorem 1, let us consider minimum invariant ellipsoid trace dependence for system error from the value of ε and α.

LMI system occurs consistent not for all α because the constraint on control resource, which is rather straightforward from technical point of view. However, it is possible to determine the interval on α (as a result of minimization procedure), where solution exists:

$$\alpha \in [0.5; 2.6]$$

.

As a result, it is possible to determine the intervals for both parameters and find a function minimum with

$$\alpha^* = 2.53, \;\; \varepsilon^* = 0.74$$

Example: AC5

The following closed-loop system parameters corresponds to above mentioned pair for $\mu = 2000$:

$$\widehat{P} = \begin{pmatrix} 1.4801 & -0.7220 & -38.3127 & 0.2563 \\ -0.7220 & 478.7808 & 11.1768 & 0.5045 \\ -38.3127 & 11.1768 & 992.8798 & -6.7034 \\ 0.2563 & 0.5045 & -6.7034 & 0.1054 \end{pmatrix},$$

$$\widehat{K_1}^T = \begin{pmatrix} -3529.6235 & -13927.3552 \\ -44.5160 & -6.1114 \\ -163.4188 & -503.9118 \\ -3988.8328 & -159.5047 \end{pmatrix}, \widehat{K_2}^T = \begin{pmatrix} -26.2802 & -666.5649 \\ -363.52 & -7.8944 \\ -2.6292 & 21.9697 \\ -331.4907 & -8.4374 \\ 17.6416 & 17.6416 \\ 3.6116 & 0.1829 \\ 3.6296 & 12.6277 \\ 331.4255 & 7.1099 \end{pmatrix},$$

and minimum invariant ellipsoid trace equals 1510.30 for $\alpha^* = 2.53$.

Example: AC5

Fig.1 depicts the dependence of resulted invariant ellipsoid trace from α and ε:

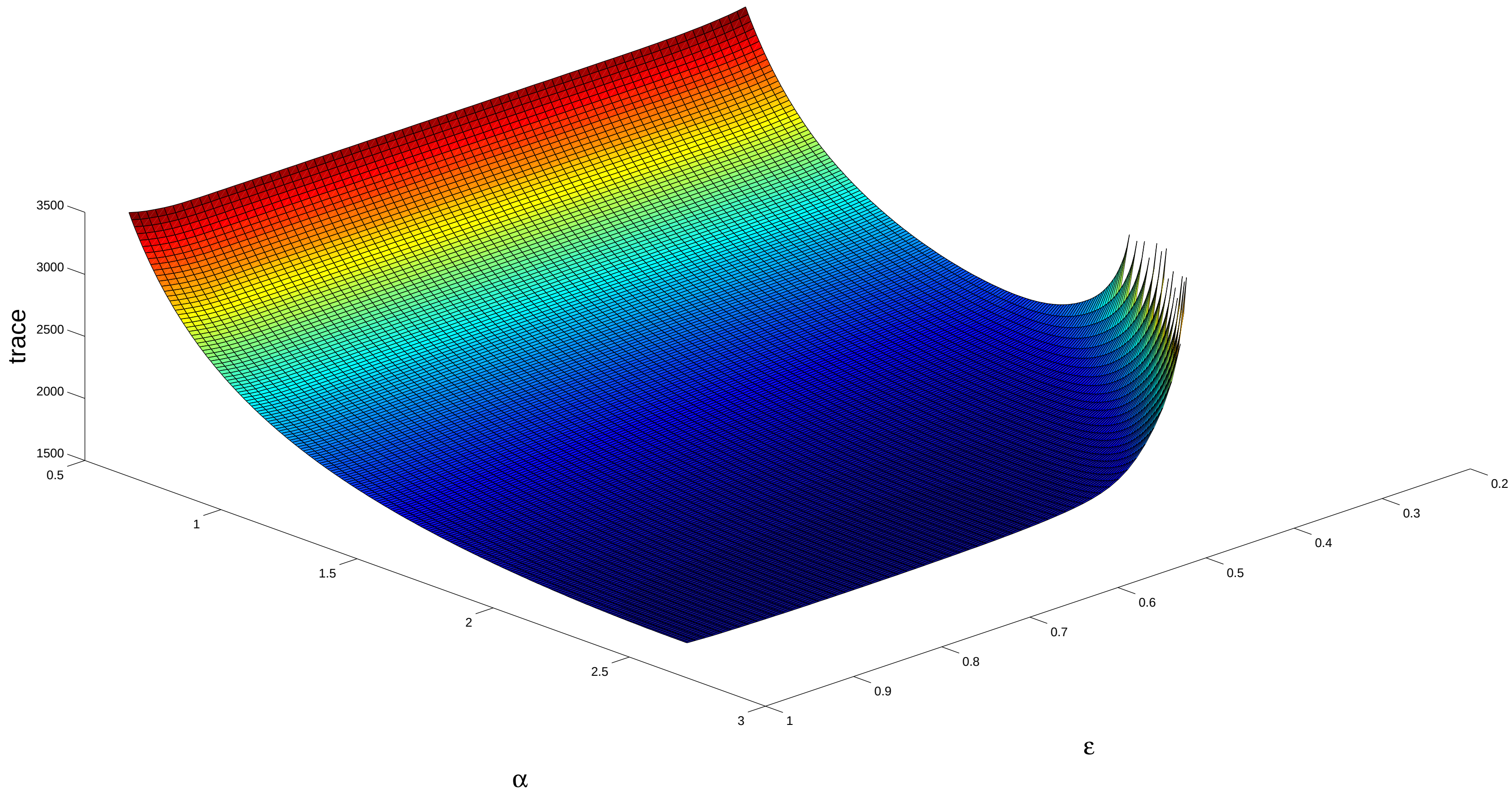
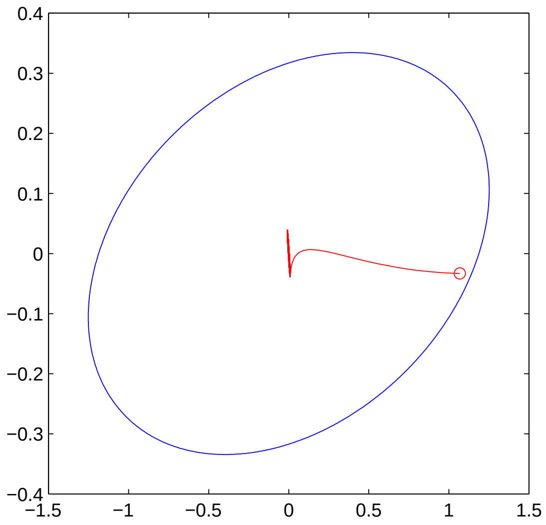


Figure 1: Invariant ellipsoid trace dependence from α and ε

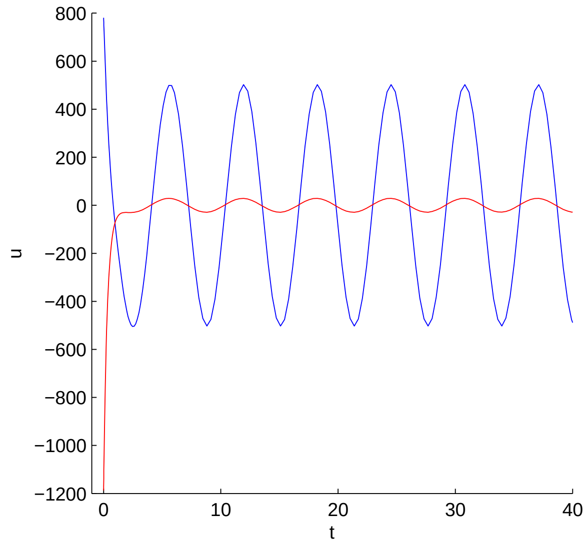
Example: AC5

Fig.2a depicts the invariant ellipsoid projection for closed-loop system error (6) and the corresponding projection of system trajectory, which start of the invariant ellipsoids border; Fig 2b. depicts the control for reference signal

$$f(t) = 1/\sqrt{2} \begin{pmatrix} 0 & \sin(t) & 0 & \cos(t) \end{pmatrix}^T .$$



a)



b)

Figure 2: Invariant ellipsoid projection, system trajectory and control

Summary

- We suggested an easy and universal approach for tracking problem solving which is based on invariant ellipsoid technique.

- Applying the invariant ellipsoid approach allows us to reformulate initial problem in terms of linear matrix inequalities and reduce searching of optimal bounding ellipsoid to semi-definite programming problem and one-dimensional convex minimization, which can be easily solved computationally.

- The method effectiveness is demonstrated on example of AC5 from COMPleib.

- Authors are planning to make further generalizations of described results: such as discrete time systems, robust statements and other tracking problem statements. Suggest approach is thought to be perspective and high-potential.

Reference list

[1] *Kalman R.* Contributions to the theory of optimal control // Boletin de la Sociedad Matematica Mexicana. 1960. №. 1. P. 102–119.

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[5] *Nazin S. A., Polyak B. T., Topunov M. V.* Dampen bounded disturbances via invariant ellipsoid method // Automatics and telemechanics. 2007. №. 3. P. 106–125.

[6] *Leibfritz F.* COMPleib: COnstraint Matrix-optimization Problem library - a collection of test examples for nonlinear semidefinite programs, control system design and related problems. Tech.-Report 2004.

Thank you for your attention!